1101 Analysis 1 Notes

Based on the lectures by Dr I Petridis

Real Numbers $\mathbb{R}$

Properties

P1 Addition's Associative Law : $a+b+c \Leftrightarrow (a+b)+c \vee a+(b+c)$

P2 Additive identity : $a+0=a$

P3 Additive inverse : $\forall a \in \mathbb{R}, \exists -a$ s.t. $a+(-a)=0$

P4 Commutative law for addition : $a+b \equiv b+a$

Subtraction : what does $a-b$ means?
Given $b \in \mathbb{R}$, P3 guarantees a $(-b)$, then $a-b = a+(-b)$.

P5 Associative law for Multiplication : $a \cdot b \cdot c \Leftrightarrow (a \cdot b) \cdot c \vee a \cdot (b \cdot c)$

P6 Multiplicative identity : $a \cdot 1 = a$ NOTE $1 \neq 0$.

P7 Multiplicative inverse : $\forall a \in \mathbb{R} \neq 0 \ \exists a^{-1}$ s.t. $a \cdot a^{-1} = 1$, also denote $a^{-1} = \frac{1}{a}$

P8 Commutative law for multiplication : $a \cdot b \equiv b \cdot a$

P9 Distributive law : $a \cdot (b+c) = a \cdot b + a \cdot c$

Inequalities

P10 Trichotomy : $\forall a, b \in \mathbb{R}$, $\exists$ 3 exclusive posibilities :
(i) $a > b$
(ii) $a = b$
(iii) $a < b$

in specific $b=0$: $\begin{cases} \text{(i) } a>0 & \text{– called } a \text{ is positive} \\ \text{(ii) } a=0 & \\ \text{(iii) } a<0 & \text{– } (-a) \text{ is positive, AKA } a \text{ is negative.} \end{cases}$

P11 If $a>0 \wedge b>0$, then $a+b>0$

P12 If $a>0 \wedge b>0$, then $a \cdot b > 0$

Manipulating inequalities

$\boxed{\text{I}}$ Transitivity : if $a>b \wedge b>c$, then $a>c$

$\boxed{\text{II}}$ If $a>b \wedge c \in \mathbb{R}$, then $a+c > b+c$

$\boxed{\text{III}}$ If $a>b \wedge c>0$, then $a \cdot c > b \cdot c$

$\boxed{\text{IV}}$ If $a>b \wedge c<0$, then $a \cdot c < b \cdot c$

Proposition If $a \neq 0$, then $a^2 > 0$. Prove.

Proof P10 Trichotomy gives either $a>0$ or $a<0$ if $a \neq 0$

Case 1 $a>0$
$a \cdot a > 0 \cdot a$ $\boxed{\text{III}}$
$a^2 > 0$

Case 2 $a<0$
$a \cdot a > 0 \cdot a$ $\boxed{\text{IV}}$
$a^2 > 0$ QED

Corollary $1 > 0$

Proof Now $1 \neq 0$ by P6
Hence $1^2 > 0$
But $1^2 = 1$
$\therefore 1 > 0$ // QED.

Example 1.6

Let $x>0$ and $y>0$. Then $x<y$ iff $x^2<y^2$. Prove.

Proof - forward.

Given $x<y$ and $x>0$, $x^2 = x\cdot x < x\cdot y$

Given $x<y$ and $y>0$, $x\cdot y < y\cdot y = y^2$

Transitivity gives $x^2 < xy < y^2 \Rightarrow x^2<y^2$

Backwards Given $x^2<y^2$,

$x^2-y^2<0$

$(x+y)(x-y)<0$

$\because x>0 \wedge y>0$

$P11 \Rightarrow x+y>0$

$\text{LEMMA} \Rightarrow \frac{1}{x+y}>0$

$x-y < \frac{0}{x+y} = 0$

$\Rightarrow x<y$ // QED.

Alternate proof By contradiction:

Assume $x^2<y^2 \Rightarrow x \geq y$ (trichotomy)

Case 1 $x=y$, then $x^2=y^2$!

Case 2 $x>y$ by part 1 of proof $\Rightarrow x^2>y^2$! // QED.

LEMMA

$\forall a \in \mathbb{R}$, a …

Proof

By contradiction

Assume $a^{-1} \leq 0$,

Now $a>0$, so

$a^{-1}\cdot a \leq 0\cdot a$

$1 \leq 0$!

$\therefore a^{-1}>0$

Backwards proof similar.

ABSOLUTE VALUES - measures the distⁿ of a $\mathbb{R}$ to 0.

Def $|a| = \begin{cases} a, & a\geq 0 \\ -a, & a<0 \end{cases}$

REMARKS $|a| \geq 0$ always true.

REMARKS [II] $|a|^2 = a^2$ **Proof:** $a\geq 0 : |a|^2 \overset{def}{=} a^2$; $a<0 : |a|^2 \overset{def}{=} (-a)^2 = a^2$

Theorem 1.15

$\forall x \in \mathbb{R}$, $-|x| \leq x \leq |x|$

Proof Case 1 $x\geq 0$, then by def $|x|=x$; Hence $-x \leq \underbrace{x \leq x}_{\text{obv} \because x=x}$

Given $x\geq 0$, we have $-x\leq 0$

$\Rightarrow -x \leq 0 \leq x \Rightarrow -x \leq x$ //

Case 2 $x<0$, then def gives $|x|=-x$; Hence $\underbrace{-(-x) \leq x}_{\text{obv} \because -(-x)=x} \leq -x$

Now $x<0$ so $-x>0$

Hence $x<0<-x \Rightarrow x<-x$ //

REMARK

$|x|<M \Leftrightarrow -M<x<M$

Proof $\because |x|\geq 0$, $M>|x| \Rightarrow M>0$

Case 1 $x\geq 0$, $|x|=x$

$|x|<M \Leftrightarrow x<M \Rightarrow -M<x<M$

Backwards

$-M<0$, $0\leq x \underset{\text{transitivity}}{\Rightarrow} -M<\cancel{x} \underset{\text{given } x<M}{\Rightarrow} -M<x<M$

Case 2 $x<0$, $|x|=-x$

$|x|<M \Leftrightarrow -x<M \Leftrightarrow x>-M$

$x<0 \wedge M>0 \Rightarrow M>x>-M$

Backwards

$-M<x \Rightarrow M>-x=|x|$ // QED

by assumptⁿ of case 2.

...F OF $\sqrt{2}$ IS IRRATIONAL

...ef x is called rational, i.e. $x \in \mathbb{Q}$ if $x = \frac{m}{n}$ with $m, n \in \mathbb{Z}$ where m, n has no common divisor (other than 1).

Proof By contradiction: Assume $\sqrt{2}$ is rational,
i.e. $\exists\, x \in \mathbb{Q}$ s.t. $x^2 = 2$

$x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}$ where $m, n \in \mathbb{Z}$

$x^2 = 2 \Rightarrow \left(\frac{m}{n}\right)^2 = 2$

$m^2 = 2n^2$ $\because n^2 \in \mathbb{Z}$, m^2 is even

$\Rightarrow m$ is even by remark 3

$\Rightarrow m = 2k \in \mathbb{Z}$ by def.

$\Rightarrow (2k)^2 = 2n^2$

$\Rightarrow 4k^2 = 2n^2$

$n^2 = 2k^2$ But $k^2 \in \mathbb{Z}$, so n^2 is even

$\Rightarrow n$ is even by remark 3.

$\because m, n$ both even, $\exists$ common divisor of 2 !.

$\therefore \sqrt{2}$ is irrational.

Triangle Inequality

$\forall a, b \in \mathbb{R}$ we have $\big||a|-|b|\big| \leq |a+b| \leq |a|+|b|$

Proof Part I: $|a+b| \leq |a|+|b|$

Consider $|a+b|^2 = (a+b)^2$ remark 2

$= a^2+b^2+2ab$

$= |a|^2+|b|^2+2ab$ remark 2

But $ab \leq |ab|$ by 1.15

$2ab \leq 2|ab|$ by $\boxed{\text{III}}$ $\because 2>0$

$|a|^2+|b|^2+2ab \leq |a|^2+|b|^2+2|a||b|$ by $\boxed{\text{III}}$ and 1.16

$\Rightarrow |a+b|^2 \leq (|a|+|b|)^2$

$\Rightarrow |a+b| \leq |a|+|b|$ by 1.6 // QED

Proof Part II: $\big||a|-|b|\big| \leq |a+b|$

Consider $|a| = |a+b-b|$

$\leq |a+b|+|-b|$ triangle

$= |a+b|+|b|$ by def

$\Rightarrow |a|-|b| \leq |a+b|$ Add $-|b|$ to both sides

Also $|b| = |b+a-a|$

$\leq |b+a|+|-a|$

$= |a+b|+|a|$

$\Rightarrow |b|-|a| \leq |a+b|$

$\because \left.\begin{matrix}|a|-|b| \\ |b|-|a|\end{matrix}\right\} \big||a|-|b|\big|$

$\therefore \big||a|-|b|\big| \leq |a+b| \leq |a|+|b|$ // QED.

Theorem $\sqrt{2}$ is irrational

i.e. $\nexists x = \frac{m}{n}$ with $m, n \in \mathbb{Z}$ s.t. $x^2 = 2$.

Def We say n is even if $\frac{n}{2} = k$ where $k \in \mathbb{Z}$.

We say n is odd if $\frac{n}{2} = k + \frac{1}{2} \Leftrightarrow n = 2k+1$ where $k \in \mathbb{Z}$

Remark 1 If n is even then n^2 is even

Proof let $n = 2k$, $n^2 = (2k)^2 = (4k^2) \in \mathbb{Z}$ $\therefore n^2$ is even

Remark 2 If n is odd then n^2 is odd

Proof let $n = 2k+1 \Rightarrow n^2 = (2k+1)^2 = (4k^2+4k)+1$ but $(4k^2+4k) \in \mathbb{Z}$, so n^2 is odd.

Remark 3 If n^2 is even then n is even

Proof Assume n^2 is even $\Rightarrow n$ is odd,

But remark 2 $\Rightarrow n^2$ is odd if n is odd !.

$\therefore$ n is even iff n^2 is even.

BOUNDS

Def a. A set S is bounded above by H means $\forall s \in S, x \leq H$

Then H is called an upper bound for set S.

b. A set S is bounded below by h means $\forall s \in S : x \geq h$

Then h is called a lower bound for set S.

Prop 1 1.5 is u.b for $S = \{x : x > 0, x^2 < 2\}$

Proof $\forall x \in S$, $x^2 < 2 < 2.25 = 1.5^2$ i.e. $x^2 < 1.5^2$

$x < 1.5 \quad \forall x \in S$.

Prop 2 $S = \{x : x > 0\}$ is unbounded above.

Proof Assume $\exists H$ s.t. $\forall x \in S$, $x \leq H$

$\because 1 > 0$, $1 \in S$, then $1 \leq H$

$0 < 2 = 1 + 1 \leq H + 1$

So $H+1 \in S$, then $\Rightarrow$ $H+1 \leq H$ $\oint$.

CONTINUUM PROPERTY (P13)

(a). If a set S is (i) non-empty and (ii) bounded above, then there exists a least u.b. called supremum, denote $\sup(S)$.

(b). If a set S is (i) non-empty and (ii) bounded below, then there exists a greatest l.b. called infimum, denote $\inf(S)$.

Remarks (a) $\forall s \in S$, $s \leq \sup(S)$ $\quad$ ($\because \sup(S)$ is an u.b.,

but $\sup(S) \leq H$ $\forall H$ as u.b. of S.

(b) $\forall s \in S$, $\inf(S) \leq s$ and $\forall h$ as l.b. of S, $h \leq \inf(S)$.

MAXIMUM & MINIMUM

Def (a). M is a <u>max</u> of S if $M \in S$ and $\forall s \in S : s \leq M$

(b) m is a <u>min</u> of S if $m \in S$ and $\forall s \in S : m \leq s$

Remarks (a) If S has a max M (denote $M = \max(S)$), then $\sup(S) = M$

(b) If S has a min m (denote $m = \min(S)$), then $\inf(S) = m$.

Prop $[1,2)$ has no maximum. Prove.

Proof Assume $\exists M \in [1,2)$ s.t. $\forall x \in [1,2) : x \leq M$;

$M \in [1,2) \Rightarrow 1 \leq M < 2$. Take $x = \frac{M+2}{2}$

(i) Prove $1 \leq x < 2$:

$M < 2$ given

$M + 2 < 4$

$\frac{M+2}{2} < 2$

$\frac{M+2}{2} = \frac{M}{2} + 1 \geq 1$

But $\frac{M+2}{2} > M$

$\Leftrightarrow M + 2 > 2M$

$\Leftrightarrow M < 2$ $\quad$ $\oint$ $\quad$ $\therefore \frac{M+2}{2} \in S$ but $x = \frac{M+2}{2} > M$

Natural Numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Properties

(i) $1 \in \mathbb{N}$

(ii) If $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$

(iii) If $n \in \mathbb{N}$, then $n \geq 1$

(iv) If $m, n \in \mathbb{N}$, $(m+n) \in \mathbb{N}$

If $m > n$, $(m-n) \in \mathbb{N}$

Archimedean Property 3.3

The set of natural numbers $\mathbb{N}$ is unbounded above.

Proof: By contradiction: Assume bounded above.

Now by (i) $1 \in \mathbb{N} \Rightarrow \mathbb{N}$ is non-empty + bounded above.

Continuum Prop $\Rightarrow \exists$ sup for $\mathbb{N}$, let $b = \sup(\mathbb{N})$.

i.e. $\forall n \in \mathbb{N}$, $n \leq b$

But $n \in \mathbb{N} \overset{(ii)}{\Rightarrow} (n+1) \in \mathbb{N} \Rightarrow (n+1) \leq b \Rightarrow n \leq b-1 \Rightarrow b-1$ is an u.b.

$\because$ $b = \sup(\mathbb{N})$, $b \leq$ all u.b. $\therefore$ $b \leq b-1 \Rightarrow 0 \geq 1$! $\therefore$ $\mathbb{N}$ is unbounded above

QED.//

Well-Ordering Principle 3.5

If S is a non-empty set of $\mathbb{N}$, it has a minimum.

Proof: Given S is non-empty and from (iii) $\forall n \in \mathbb{N} \Rightarrow n \geq 1$ i.e. bounded below by 1.

Continuum prop gives a greatest lower bound (inf). Let $\inf(S) = b$

i.e. $\forall m \in \mathbb{N}$, $b \leq m$.

Case 1: $b = m$, then $m \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, $n \geq b = m \Rightarrow m = \min(S)$ // min exists.

Case 2: $b < m$

$\Rightarrow m$ is not a l.b. $\because$ $b = \inf(S)$.

$\Rightarrow \exists n \in \mathbb{N}$ s.t. $b \leq n < m$

Also $\because$ $b = \inf(S) \Rightarrow b+1$ is not a lower bound. i.e. $\exists k \in \mathbb{N}$ s.t. $k < b+1$.

$\because$ $m \in \mathbb{N}$, take $k = m$.

We have $b \leq n < m < b+1$.

But $b \leq n \Rightarrow -b \geq -n$

and $b+1 > m$ (+)

$$b+1-b = 1 > m-n$$

i.e. $m - n < 1$ ——— #

However given $m > n$, (iv) gives:

$(m-n) \in \mathbb{N}$ and from (iii):

$(m-n) \geq 1$! with #

Hence case 2 invalid.

i.e. $\min(S)$ exists and $\min(S) = \inf(S)$ // QED.

PRINCIPLE OF MATHEMATICAL INDUCTION – PROOF

Let $P(n)$ be a proposition with $n \in \mathbb{N}$.
Given $P(1)$ is true and $P(n) \Rightarrow P(n+1)$, then $P(n)$ is true $\forall n \in \mathbb{N}$

Proof By contradiction: Assume given conditions, $\exists n \in \mathbb{N}$ s.t. $P(n)$ is false.
Define $S = \{n : P(n) \text{ is false}\}$
Given $P(1)$ is true $\Rightarrow 1 \notin S$.
Also, assumption gives ~~$m \in S$ i.e.~~ S is non-empty.
Well-ordering principle supplys $\exists \min(S)$, let $m = \min(S)$.
Then $m-1 \notin S$ $\because m = \min(S)$. i.e. $P(m-1) \rightarrow$ true.
But $P(m-1)$ true $\Rightarrow P(m)$ true [by conditions]
$\lightning$ $\because m = \min(S) \Rightarrow m \in S$ i.e. $P(m)$ is false $\lightning$.
$\therefore$ Inductive principle must be true // QED.

Existance of $\sqrt{2}$ i.e. $\exists a \in \mathbb{R}$ s.t. $a^2 = 2$.

Proof Define $S = \{x : 0 < x \text{ and } x^2 < 2\}$
Consider $x = 1$, $1 > 0$ and $1^2 = 1 < 2$,
hence $1 \in S \Rightarrow S$ is Non$\varnothing$.
$\forall x \in S$, $x^2 < 2 < 4 \Rightarrow x^2 < 4 \underset{1.6}{\Rightarrow} x < 2$ $\forall x \in S$, i.e. 2 is an u.b. of S.
$\therefore S$ is non-$\varnothing$ & bounded above, continuum prop $\Rightarrow \sup(S)$ exists. let $\sup(S) = a$.

By Trichotomy $a^2 > 2 \vee a^2 = 2 \vee a^2 < 2$

Case 1 $a^2 < 2$
Consider $y = a + \frac{1}{n}$, $x^2 = (a+\frac{1}{n})^2 = a^2 + \frac{2a}{n} + \frac{1}{n^2} \leq a^2 + \frac{2a+1}{n} < a^2 + \frac{2\cdot 2+1}{n} = a^2 + \frac{5}{n}$
$\left(\begin{array}{c} n \geq 1 \\ n^2 \geq n \\ \frac{1}{n^2} \leq \frac{1}{n} \end{array}\right)$
$\because$ Archimedean prop says $\mathbb{N}$ is unbounded above,
$\exists n \in \mathbb{N}$ s.t. $n > \frac{5}{2-a^2} \Rightarrow \frac{1}{n} < \frac{2-a^2}{5} \Rightarrow a^2 + \frac{5}{n} < 2$
Hence $\exists n$ s.t $y^2 < 2$, also $\because a = \sup(S)$ & $1 \in S$, $a \geq 1 \Rightarrow 1 \leq a \leq a + \frac{1}{n} = y$
i.e. $y > 0$ & $y < 2 \Rightarrow x \in S$.
BUT $y = a + \frac{1}{n} > a$ while $a = \sup(S) = $ u.b of S $\lightning$.

Case 2 $a^2 > 2$
Consider $z = a - \frac{1}{n}$, $z^2 = (a-\frac{1}{n})^2 = a^2 - \frac{2a}{n} + \frac{1}{n^2} > a^2 - \frac{2a}{n}$
And Archimedean prop provides $n \in \mathbb{N}$ s.t. $n > \frac{2a}{a^2-2} \Rightarrow \frac{1}{n} < \frac{a^2-2}{2a} \Rightarrow a^2 - \frac{2a}{n} > 2$
Hence $\exists n$ s.t. $z^2 > 2$.
But $\forall x \in S$, $x^2 < 2$ hence $x^2 < z^2 \Leftrightarrow x < z$ $\forall x \in S$. i.e. z is an u.b.
But $z = a - \frac{1}{n} < a = $ least u.b. $\lightning$.

$\therefore$ Only case valid is **Case 3** $a^2 = 2$. Hence $\sqrt{2}$ exists. // QED.

THEOREM

If $\zeta > 0$ and the set S is non-$\emptyset$ and bounded above, then:

the set $\zeta S = \{\zeta x : x \in S\}$ is also bounded above and $\sup(\zeta S) = \zeta \sup$

Proof S is non-$\emptyset$ + bounded above $\underset{\text{cont. prop.}}{\Rightarrow}$ $\exists \sup(S)$ s.t. $\forall x \in S$, $x \leq \sup(S)$.

Given $\zeta > 0$, $\forall x \in S$: $\zeta x \leq \zeta \sup(S)$ —— $*$

Hence $\zeta \sup(S)$ is an u.b. for $\forall \zeta x$ i.e. ζS is bounded above.

Continuum prop $\Rightarrow$ $\exists \sup(\zeta S)$ as least u.b. for ζS.

$\Rightarrow \sup(\zeta S) \leq \zeta \sup(S)$ —— ①

Also $\forall \zeta x \in \zeta S$, $\zeta x \leq \sup(\zeta S)$

$\Rightarrow x \leq \dfrac{\sup(\zeta S)}{\zeta} \quad \forall x \in S$.

$\Rightarrow \dfrac{\sup(\zeta S)}{\zeta}$ is an u.b. for set S.

$\because \sup(S)$ is least u.b., then

$\sup(S) \leq \dfrac{\sup(\zeta S)}{\zeta}$

$\Rightarrow \zeta \sup(S) \leq \sup(\zeta S)$ —— ②

① $\wedge$ ② both true $\Rightarrow$ $\zeta \sup(S) = \sup(\zeta S)$ // QED.

~~S~~EQUENCES

Def: A seq is an ordered list of $\mathbb{R}$: $\langle x_n \rangle_{n=1,2,\ldots}$ $\langle x_n \rangle : x \ldots x$

Convergence of sequences

Def We say a seq $\langle x_n \rangle$ converges to the number l if:

$$\left\{ \begin{array}{l} \text{Given } \varepsilon > 0 \text{ I can find } N \text{ s.t. whenever } n > N \text{, we have } |x_n - l| < \varepsilon \\ \text{i.e. } \forall \varepsilon > 0 \ \exists N \text{ s.t. } n > N \Rightarrow |x_n - l| < \varepsilon \end{array} \right\}$$

We write $\lim_{n\to\infty} x_n = l$ or $x_n \to l$

Example Prove $\lim_{n\to\infty} x_n = 1$, $x_n = \frac{n^2-1}{n^2+1}$

Take $N = \sqrt{\frac{2}{\varepsilon} - 1}$, Now:

Given $\varepsilon > 0$

$n > N \Rightarrow n > \sqrt{\frac{2}{\varepsilon} - 1} \Rightarrow n^2 > \frac{2}{\varepsilon} - 1 \Rightarrow n^2 + 1 > \frac{2}{\varepsilon} \Rightarrow \varepsilon > \frac{2}{n^2+1}$

$\Rightarrow \left| \frac{-2}{n^2+1} \right| < \varepsilon \Rightarrow \left| \frac{n^2 - 1 - n^2 - 1}{n^2+1} \right| < \varepsilon \Rightarrow \left| \frac{n^2-1}{n^2+1} - \frac{(n^2+1)}{n^2+1} \right| < \varepsilon \Rightarrow \left| \frac{n^2-1}{n^2+1} - 1 \right| < \varepsilon$ //

Hence $\forall \varepsilon > 0$, $\exists N = \sqrt{\frac{2}{\varepsilon} - 1}$ s.t. $n > N \Rightarrow |x_n - 1| < \varepsilon$, $(\forall \varepsilon > 0)$.

$\therefore \lim_{n\to\infty} x_n = 1$ // QED.

~~Con~~ Theorem

Let $\langle x_n \rangle$ be the constant sequence, i.e. $x_n = k \ \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} k = k$.

Proof Take N as any $\mathbb{N}$, say $N = 0$:

we have $n > 0$ & $\varepsilon > 0 \Leftrightarrow \varepsilon > |k - k| \Leftrightarrow |x_n - k| < \varepsilon$, $\forall \varepsilon > 0$. QED //

COMBINATION THEOREMS

let $\langle x_n \rangle$ & $\langle y_n \rangle$ be two convergent seq with $\lim_{n\to\infty} x_n = l$ & $\lim_{n\to\infty} y_n = m$, then:

(a) the seq $\langle x_n + y_n \rangle$ is convergent and $\lim_{n\to\infty} (x_n + y_n) = l + m$

(b) the seq $\langle \lambda x_n \rangle$ where $\lambda \in \mathbb{R}$ is convergent and $\lim_{n\to\infty} (\lambda x_n) = \lambda \cdot l$

(c) the seq $\langle x_n \cdot y_n \rangle$ is convergent and $\lim_{n\to\infty} (x_n y_n) = l \cdot m$

(d) the seq $\left\langle \frac{x_n}{y_n} \right\rangle$ is convergent if $m \neq 0$ and $\lim_{n\to\infty} \left(\frac{x_n}{y_n} \right) = \frac{l}{m}$

Proof (b)

Case 1 $\lambda = 0$, $\lim_{n\to\infty} (\lambda \cdot x_n) = \lim_{n\to\infty} 0 = 0$

Case 2 $\lambda \neq 0$. Given $\lim_{n\to\infty} x_n = l$ prove $\lim_{n\to\infty} \lambda x_n = \lambda l$:

$\lim_{n\to\infty} x_n = l \Rightarrow (\forall \varepsilon > 0 \ \exists N \text{ s.t. } n > N \Rightarrow |x_n - l| < \varepsilon)$

$$\left. \begin{array}{l} \text{Now } \frac{\varepsilon}{|\lambda|} > 0 \text{ if } \varepsilon > 0, \\ \text{def} \Rightarrow \left(n > N \Rightarrow |x_n - l| < \frac{\varepsilon}{|\lambda|} \right. \\ \qquad \Rightarrow |\lambda||x_n - l| < \varepsilon \\ \qquad \Rightarrow |\lambda x_n - \lambda l| < \varepsilon \Big) \end{array} \right\} \begin{array}{l} \Rightarrow \forall \varepsilon > 0 \ \exists N \text{ s.t. } \\ \quad n > N \Rightarrow |\lambda x_n - \lambda l| < \varepsilon \\ \\ \Rightarrow \lim_{n\to\infty} (\lambda x_n) = \lambda \cdot l \text{ // QED.} \end{array}$$

Cauchy-Schwarz Inequality

Let $a_1, a_2, \dots, a_n$ & $b_1, b_2, \dots, b_n$ be any $\mathbb{R}$,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n}(a_i)^2 \cdot \sum_{i=1}^{n}(b_i)^2$$

Proof Consider $\sum_{i=1}^{n}(a_i x+b_i)^2 \geq 0$ true since any $a \in \mathbb{R}$ satisfy $a^2 \geq 0$

$$\Rightarrow (a_1x+b_1)^2+(a_2x+b_2)^2+\dots+(a_nx+b_n)^2 \geq 0$$

$$\Rightarrow (a_1^2x^2+2a_1b_1x+b_1^2)+(a_2^2x^2+2a_2b_2x+b_2^2)+\dots+(a_n^2x^2+2a_nb_nx+b_n^2) \geq 0$$

$$(a_1^2+a_2^2+\dots+a_n^2)x^2+2(a_1b_1+a_2b_2+\dots+a_nb_n)x+(b_1^2+b_2^2+\dots+b_n^2) \geq 0$$

$$\Rightarrow \Delta \leq 0$$

$$\Rightarrow [2(a_1b_1+a_2b_2+\dots+a_nb_n)]^2-4(a_1^2+a_2^2+\dots+a_n^2)(b_1^2+b_2^2+\dots+b_n^2) \leq 0$$

$$(a_1b_1+a_2b_2+\dots+a_nb_n)^2 \leq (a_1^2+a_2^2+\dots+a_n^2)(b_1^2+b_2^2+\dots+b_n^2)$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \cdot \sum_{i=1}^{n} b_i^2 \quad // \text{ QED.}$$

Combination Theorem (a) – Proof

Given $\lim x_n = l$ & $\lim y_n = m$, $\lim(x_n+y_n) = l+m$.

Proof $\lim x_n = l \Rightarrow (\forall \varepsilon>0\ \exists N_1 \text{ s.t. } n>N_1 \Rightarrow |x_n-l|<\varepsilon)$, and

$\lim y_n = m \Rightarrow (\forall \varepsilon>0\ \exists N_2 \text{ s.t. } n>N_2 \Rightarrow |y_n-m|<\varepsilon)$.

~~Take~~ Let $N = \max(N_1, N_2)$, then $\forall n>N \Rightarrow (|x_n-l|<\varepsilon \wedge |y_n-m|<\varepsilon)$, true $\forall \varepsilon>0$.

$\because \frac{\varepsilon}{2}>0$ also, $n>N \Rightarrow |x_n-l|<\frac{\varepsilon}{2} \wedge |y_n-m|<\frac{\varepsilon}{2}$ true#.

Consider $|(x_n+y_n)-(l+m)|$

$$= |(x_n-l)+(y_n-m)|$$

$$\leq |x_n-l|+|y_n-m|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{by \#}$$

$$= \varepsilon$$

$\therefore n>N \Rightarrow |(x_n+y_n)-(l+m)|<\varepsilon$, $\forall \varepsilon>0$ $\therefore \lim(x_n+y_n) = l+m$

Sandwich Theorem

Given $y_n \le x_n \le z_n$ and $\lim y_n = l = \lim z_n$, then $\lim x_n = l$ also.

Proof $\lim y_n = l \Rightarrow (\forall \varepsilon > 0\ \exists N_1$ s.t. $n > N_1 \Rightarrow |y_n - l| < \varepsilon)$

$\lim z_n = l \Rightarrow (\forall \varepsilon > 0\ \exists N_2$ s.t. $n > N_2 \Rightarrow |z_n - l| < \varepsilon)$

Choose/Let $N = \max(N_1, N_2)$,

then $\forall n > N$: $|y_n - l| < \varepsilon \ \wedge\ |z_n - l| < \varepsilon$

$\Rightarrow l - \varepsilon < y_n < l + \varepsilon \ \wedge\ l - \varepsilon < z_n < l + \varepsilon$ true.

Given $y_n \le x_n \le z_n$

$\Rightarrow l - \varepsilon < y_n \le x_n \le z_n < l + \varepsilon$

$\Rightarrow l - \varepsilon < x_n < l + \varepsilon \quad \forall n > N$

$\Rightarrow (n > N \Rightarrow |x_n - l| < \varepsilon)\ \forall \varepsilon > 0 \Rightarrow \lim x_n = l$ // QED.

Monotonic Sequences

Def A seq $\langle x_n \rangle$ is called increasing if $x_n \le x_{n+1}$, $n \in \mathbb{N}$

A seq $\langle x_n \rangle$ is called decreasing if $x_n \geq x_{n+1}$, $n \in \mathbb{N}$

A seq $\langle x_n \rangle$ is called strictly increasing if $x_n > x_{n+1}$, $n \in \mathbb{N}$

A seq $\langle x_n \rangle$ is called strictly decreasing if $x_n < x_{n+1}$, $n \in \mathbb{N}$

Theorem

(a) If $\langle x_n \rangle$ is increasing and bounded above, $\langle x_n \rangle$ converges and converges to its sup.

(b) If $\langle x_n \rangle$ is decreasing and bounded below, $\langle x_n \rangle$ converges and converges to its inf.

Proof (a) let $\sup(x_n) = b$, Given increasing $(x_n \le x_{n+1})$ and bounded above $(x_n \le M\ \forall n \in \mathbb{N})$, aim to prove $\lim x_n = b$. /// 1 $\because$ bounded above, Continuum property $\Rightarrow \sup(x_n)$ exists, let $b = \sup(x_n)$.

2 Now $\because$ b is the least ~~for~~ u.b., $\exists N$ s.t. $x_N > b - \varepsilon \quad \forall \varepsilon > 0$

3 But b is an u.b. so $x_n \le b$. 4 increasing $\Rightarrow (n > N \Rightarrow x_n \geq x_N)$.

Combining we have $b - \varepsilon < x_N \le x_n \le b < b + \varepsilon$, $\forall \varepsilon > 0$ and $n > N$.

$\Rightarrow b - \varepsilon < x_n < b + \varepsilon \Rightarrow |x_n - b| < \varepsilon$, $\forall \varepsilon > 0$ and $n > N$.

$\therefore \lim x_n = b = \sup(x_n)$ // QED.

Proof (b)

let $\langle x_n \rangle$ be decreasing and bounded below by m, then $\langle -x_n \rangle$ is increasing and bounded above by $-m$.

By (a), $\lim(-x_n) = \sup(-x_n)$

$-\lim x_n = \sup(-x_n)$ combination (b)

$\lim x_n = -\sup(-x_n)$ #

$= \inf(x_n)$ // QED

\# $-\sup(-S) = \inf(S)$, prove later.

Theorem

Let S be a set of real numbers and bounded below and def $-S = \{-s, s$

then $-S$ is bounded above and $\sup(-s) = -\inf(s)$.

Proof

S is bounded below, continuum prop $\Rightarrow \exists \inf(s)$ s.t. $s \geq \inf(s)$ $\forall s \in S$.

then $-s \leq -\inf(s)$

$\Rightarrow -\inf(s)$ is an u.b. for $-s$.

$\because -\inf(s) \geq$ least u.b of $-S = \sup(-s)$ $\therefore -\inf(s) \geq \sup(-s)$ —— ①

Also $-S$ bounded above $\Rightarrow -s \leq \sup(-S)$ $\forall -s \in -S$

$\Rightarrow s \geq -\sup(-s)$ i.e. $-\sup(-s)$ is l.b. for set S,

$\Rightarrow \inf(S) \geq -\sup(-s)$ $\because \inf(s) =$ greatest l.b.

$\Rightarrow -\inf(s) \leq \sup(-s)$ —— ②

Combine ① & ② gives $-\inf(s) = \sup(-s)$ // QED.

Theorem

If $\langle x_n \rangle$ is convergent, then its limit is unique.

i.e. Assume $\lim x_n = l$ & $\lim x_n = m$, then $l = m$.

Proof

By contradiction: Assume $l \neq m$, then $|l-m| > 0$

Take $\varepsilon = \frac{|l-m|}{2}$,

$\lim x_n = l \Rightarrow (\exists N_1 \text{ s.t. } n > N_1 \Rightarrow |x_n - l| < \frac{|l-m|}{2})$

$\lim x_n = m \Rightarrow (\exists N_2 \text{ s.t. } n > N_2 \Rightarrow |x_n - m| < \frac{|l-m|}{2})$

Choose $N = \max(N_1, N_2)$,

Then for $n > N$ we have $\left(|x_n - l| < \frac{|l-m|}{2}\right) \wedge \left(|x_n - m| < \frac{|l-m|}{2}\right)$ true $\forall \varepsilon = \frac{|l-m|}{2} > 0$

Consider: $|x_n - l| + |x_n - m| < \frac{|l-m|}{2} + \frac{|l+m|}{2}$

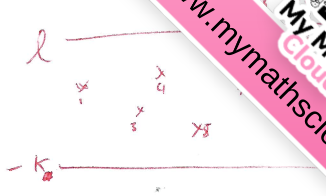
$|x_n - l + m - x_n| \underset{\text{tringle}}{\leq} |x_n - l| + |m - x_n| < |l - m|$

$\Rightarrow |m - l| = |l - m| < |l - m|$! $\therefore l \neq m$

$\Rightarrow \lim x_n$ is unique. //

Theorem

If $\langle x_n \rangle$ is convergent, it is bounded. (above + below)



Proof $\langle x_n \rangle$ is convergent $\Rightarrow \lim x_n$ exists. let $\lim x_n = l$.

i.e. $\forall \varepsilon > 0 \; \exists N$ s.t. $n > N \Rightarrow |x_n - l| < \varepsilon$ (def of lim)

Take $\varepsilon = 1$, $n > N \Rightarrow |x_n - l| < \varepsilon = 1$

$\Rightarrow -1 < x_n - l < 1$

$\Rightarrow l - 1 < x_n < l + 1$

Let $M = \max(x_1, x_2, \dots, x_N, l+1)$, and

$m = \min(x_1, x_2, \dots, x_N, l-1)$.

Proposition (a) M is an u.b. for $\langle x_n \rangle$ and (b) m is a l.b.

Proof (a): M is u.b. $\Rightarrow x_n \leq M, \forall n \in \mathbb{N}$.

Consider $n > N$, def of lim gives $x_n < l+1 \leq M = \max(\dots)$

for $n \leq N$, x_n appears in $\{x_1, x_2, \dots, x_N\}$,

$\because M = \max(x_1, x_2, \dots, x_N, l+1)$, $M \geqslant x_n$

Proof (b): m is l.b. $\Rightarrow x_n \geqslant m, \forall n \in \mathbb{N}$

Consider $n > N$, def gives $x_n > l - 1 \geqslant m = \min(\dots)$

for $n \leq N$, x_n appears in $(x_1, x_2, \dots, x_N)$,

Then all $x_n \geqslant m = \min(\dots)$

$\therefore$ we have $m \leqslant x_n \leqslant M \; \forall n \in \mathbb{N}$, i.e. $\langle x_n \rangle$ is bounded by M, m. // QED.

Combination Theorem (c) – Proof

Given $\lim x_n = l$ and $\lim y_n = m$, $\lim(x_n y_n) = lm$.

Proof Consider $|x_n y_n - lm| = |x_n y_n - l y_n + l y_n - lm|$

$= |y_n(x_n - l) + l(y_n - m)|$

$\leqslant |y_n(x_n - l)| + |l(y_n - m)|$

$= |y_n||x_n - l| + |l| \cdot |y_n - m|$

Given $x_n \to l$ & $y_n \to m$, we have $x_n - l \to 0$ & $y_n - m \to 0$

Also $|y_n|$ is convergent $\underset{\text{theorem above}}{\Rightarrow}$ y_n is bounded, say $|y_n| \leqslant K \; \forall n \in \mathbb{N}$.

then: $0 \leqslant |x_n y_n - lm| \leqslant K \underset{\downarrow 0}{|x_n - l|} + |l| \cdot \underset{\downarrow 0}{|y_n - m|}$

LHS $= 0$, RHS $\to 0$

Sandwich theorem $\Rightarrow x_n y_n - lm \to 0$ as $n \to \infty$

i.e. $x_n y_n \to lm$ as $n \to \infty$ // QED.

Combination Theorem (d) - Proof

Given $\lim x_n = l$ & $\lim y_n = m \neq 0$, then $\lim \frac{x_n}{y_n} = \frac{l}{m}$. i.e. $\frac{x_n}{y_n} \to \frac{l}{m}$.

Lemma

If $\lim y_n = m \neq 0$, then $\exists N$ s.t. $n > N \Rightarrow |y_n| > \frac{|m|}{2}$. (e.g. $\lim y_n = 2$, then $\exists N$ s.t. $n > N$

Proof $\lim y_n = m \Rightarrow (\forall \varepsilon > 0\ \exists N$ s.t. $n > N \Rightarrow |y_n - m| < \varepsilon)$

Consider $\varepsilon = \frac{|m|}{2} > 0$, $\exists N$ s.t. $n > N \Rightarrow |y_n - m| < \frac{|m|}{2}$

$$|m| - |y_n| \underbrace{\leq}_{a \leq |a|} \big||-m| - |y_n|\big| \underbrace{\leq}_{\text{triangle}} |-m + y_n| < \frac{|m|}{2}$$

$$\Rightarrow \quad -|y_n| < \frac{|m|}{2} - |m| = -\frac{|m|}{2}$$

$$|y_n| > \frac{|m|}{2} \quad \text{for } n > N \ // \ \text{Lemma proved.}$$

Proof (d) Consider $\left|\frac{x_n}{y_n} - \frac{l}{m}\right| = \left|\frac{x_n \cdot m - l \cdot y_n}{y_n \cdot m}\right| = \frac{|m \cdot x_n - l \cdot y_n|}{|y_n| \cdot |m|} < \frac{|m x_n - l y_n|}{|m|} \cdot \frac{2}{|m|} = \frac{2}{|m|^2} \cdot |m \cdot x_n - l \cdot y_n|$

Given $n > N$:
(N from lemma)

Lemma
$n > N \Rightarrow |y_n| > \frac{|m|}{2}$
$\Rightarrow \frac{1}{|y_n|} < \frac{2}{|m|}$

But $x_n \to l$ & $y_n \to m$,

then $m \cdot x_n \to m \cdot l$ & $l \cdot y_n \to m \cdot l$ (combination (b))

$\Rightarrow |m \cdot x_n - l \cdot y_n| \to |ml - ml| = 0$, RHS $= \frac{2}{|m|^2} |m x_n - l \cdot y_n| \to 0$

LHS $> 0 \to 0$

$\therefore \frac{x_n}{y_n} - \frac{l}{m} \to 0$, $\Rightarrow \frac{x_n}{y_n} \to \frac{l}{m}$ // QED.

Theorem 4.23

(a) If $x_n < a$ $\forall n \in \mathbb{N}$ and $\lim x_n = l$, then $l \leq a$.

(b) If $x_n > b$ $\forall n \in \mathbb{N}$ and $\lim x_n = l$, then $l \geq b$.

Proof (a) By contradiction: Assume $\left\{\begin{array}{l} x_n < a \\ \lim x_n = l \end{array}\right\} \Rightarrow l > a$;

$\lim x_n = l \Rightarrow (\forall \varepsilon > 0\ \exists N$ s.t. $n > N \Rightarrow |x_n - l| < \varepsilon)$

If $l > a$ true, $l - a > 0$. Consider $\varepsilon = l - a > 0$,

then $n > N \Rightarrow |x_n - l| < l - a$

But $l - x_n \leq |l - x_n| < l - a$

$\Rightarrow -x_n < -a \Rightarrow x_n > a$ BUT Assumption $x_n < a$ ↯.

$\therefore l > a$ false, $l \leq a$ true // QED.

Proof (b) Given $x_n > b$ and $\lim x_n = l$ i.e. $x_n \to l$

$-x_n < -b$ $\quad \lim -x_n = (-l)$ $\quad -x_n \to -l$

By (a), $(-l) \leq (-b)$

$\Rightarrow l \geq b$ // QED.

1-GM INEQUALITY

Let $a_1, a_2, \cdots, a_n \geq 0$ and define $AM = \dfrac{a_1+a_2+\cdots+a_n}{n}$ and $GM =$

then $AM \geq GM$ i.e. $\dfrac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$ //.

Lemma

Let $n \geq 2$ and $b_1, b_2, \cdots, b_n$ be positive numbers with $b_1 \cdot b_2 \cdots b_n = 1$,

then $b_1 + b_2 + \cdots + b_n \geq n$.

Proof By induction.

$P(1)$: $n=2$, $b_1 \cdot b_2 = 1$ given. Then $b_1+b_2 = b_1 + \frac{1}{b_1} = \frac{b_1^2+1}{b_1}(-2+2) = \frac{b_1^2-2b_1+1}{b_1}+2 = \frac{(b_1-1)^2}{b_1}+2 \geq 2$

Assume $P(n)$ is true: $b_1 \cdot b_2 \cdots b_n = 1 \Rightarrow b_1+b_2+\cdots+b_n \geq n$ *

Aim to prove $P(n+1)$: $b_1 \cdot b_2 \cdots b_n \cdot b_{n+1} = 1 \Rightarrow b_1+b_2+\cdots+b_n+b_{n+1} \geq n+1$

Case 1 $b_1 = b_2 = 1, \cdots = b_{n+1} = 1$, then $b_1+b_2+\cdots+b_n+b_{n+1} = n+1$ ✓

Case 2 $\exists i \in \mathbb{N}$ s.t. $b_i > 1$, then $\exists k$ s.t. $b_k < 1$

Consider / let $n = i$, i.e. $b_i = b_n > 1$ and take $b_{n+1} < 1$

We have $b_n - 1 > 0$

and $1 - b_{n+1} > 0$

multiply gives $(b_n-1)(1-b_{n+1}) = b_n + b_{n+1} - b_n b_{n+1} - 1 > 0$

$b_n + b_{n+1} > 1 + b_n \cdot b_{n+1}$ —— ①

Now, Aim LHS $\Rightarrow$ RHS.

LHS: $b_1 \cdot b_2 \cdots b_{n-2} \cdot b_{n-1} (b_n \cdot b_{n+1}) = 1$ ← take as 1 term

$\Rightarrow b_1+b_2+\cdots+b_{n-2}+b_{n-1}+(b_n \cdot b_{n+1}) \geq n$ * from inductive assumption

Also have: $b_n + b_{n+1} > 1 + b_n \cdot b_{n+1}$ —— from ①

Adding gives: $b_1+b_2+\cdots+b_n \cdot b_{n+1} + b_n + b_{n+1} > n+1+b_n \cdot b_{n+1}$

$\Rightarrow b_1+b_2+\cdots+b_n+b_{n+1} > n+1$

$\therefore$ By induction we have $b_1+b_2+\cdots+b_n \geq n$ // given $b_1 \cdot b_2 \cdots b_n = 1$ //.

Proof of AM-GM

Case 1 $\exists i$ s.t. $a_i = 0$, then $GM = 0$ and $AM = \dfrac{a_1+a_2+\cdots+0+\cdots+a_n}{n} \geq 0$ $\therefore AM \geq GM$.

Case 2 $a_1, a_2, \cdots, a_n > 0$,

Define $G = \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdots a_n}$

and let $b_1 = \frac{a_1}{G}, b_2 = \frac{a_2}{G}, \cdots, b_n = \frac{a_n}{G}$;

$$b_1 \cdot b_2 \cdots b_n = \frac{a_1}{G} \cdot \frac{a_2}{G} \cdots \frac{a_n}{G} = \frac{a_1 \cdot a_2 \cdots a_n}{G^n} = \frac{a_1 \cdot a_2 \cdots a_n}{\left(\sqrt[n]{a_1 \cdot a_2 \cdots a_n}\right)^n} = 1$$

then lemma gives $b_1+b_2+\cdots+b_n \geq n$

$$\frac{a_1}{G}+\frac{a_2}{G}+\cdots+\frac{a_n}{G} \geq n$$

$$\frac{a_1+a_2+\cdots+a_n}{G} \geq n$$

$$AM = \frac{a_1+a_2+\cdots+a_n}{n} \geq G = \sqrt[n]{a_1 a_2 \cdots a_n} = GM$$ // QED

Application 1

let $x_n = \left(1+\frac{1}{n}\right)^n$, Prove that x_n is increasing.

Proof i.e. To prove $x_n \geq x_{n-1}$, i.e. $\left(1+\frac{1}{n}\right)^n \geq \left(1+\frac{1}{n-1}\right)^{n-1}$

let $a_1 = a_2 = \cdots = a_{n-1} = 1+\frac{1}{n-1}$ and $a_n = 1$

$$AM = \frac{a_1+a_2+\cdots+a_{n-1}+a_n}{n} = \frac{\left(1+\frac{1}{n-1}\right)(n-1)+1}{n} = \frac{(n-1+1)+1}{n} = \frac{n+1}{n} = 1+\frac{1}{n}$$

$$GM = \sqrt[n]{a_1 a_2 \cdots a_{n-1} a_n} = \left[\left(1+\frac{1}{n-1}\right)^{n-1}\times 1\right]^{\frac{1}{n}} = \left(1+\frac{1}{n-1}\right)^{\frac{n-1}{n}}$$

But $GM \leq AM$

$\therefore \left(1+\frac{1}{n-1}\right)^{\frac{n-1}{n}} \leq 1+\frac{1}{n}$

$\left(1+\frac{1}{n-1}\right)^{n-1} \leq \left(1+\frac{1}{n}\right)^n$

$\Rightarrow \quad x_{n-1} \leq x_n$ // QED. $\langle x_n \rangle$ is increasing.

– END OF SEQUENCES –

SUBSEQUENCES

Def Let $\langle n_r \rangle_{r=1,2,\dots}$ be a strictly increasing sequence with $n_r \in \mathbb{N}$.
Then $\langle x_{n_r} \rangle_{r=1,2,\dots}$ is called a subsequence of $\langle x_n \rangle_{n=1,2,\dots}$.

Theorem

Given a convergent seq $\langle x_n \rangle_{n=1,2,\dots}$ with $\lim_{n\to\infty} x_n = l$, its subseq $\langle x_{n_r} \rangle$ is also convergent and $\lim_{r\to\infty} x_{n_r} = l$.

i.e. All subseq of an convergent seq tends to the same limit.

Remark If $\exists$ 2 subseq of a seq with different limit, the original seq is not convergent.

Proof **LEMMA** let $\langle n_r \rangle$ $n_r \in \mathbb{N}$ be a strictly increasing seq, then $n_r \geqslant r \quad \forall r \in \mathbb{N}$.

Induction: $P(r)$: $n_r \geqslant r$

$P(1)$ $n_1 \geqslant 1$ true since $n_r \in \mathbb{N} \geqslant 1$.

Assume $P(k)$ true: $n_k \geqslant k$

$P(k+1)$: $n_{k+1} \underset{\text{strictly increasing}}{>} n_k \underset{\text{assumption}}{\geqslant} k \Rightarrow n_{k+1} > k$

$\Rightarrow n_{k+1} \geqslant k+1 \quad \because n_r \in \mathbb{N}$

Lemma proved.

Proof of Theorem

Given $\lim_{n\to\infty} x_n = l \overset{\text{"def"}}{\Rightarrow} (\forall \varepsilon > 0\ \exists N \text{ s.t. } n > N \Rightarrow |x_n - l| < \varepsilon)$

To prove: $\lim_{r\to\infty} x_{n_r} = l$ i.e. $(\forall \varepsilon > 0\ \exists R \text{ s.t. } r > R \Rightarrow |x_{n_r} - l| < \varepsilon)$

Take $N = R$, we have $\forall \varepsilon > 0$; $\exists R$ s.t. $r > R \Rightarrow r > N$ but $n_r \geqslant r$ by lemma

$n_r \geqslant r > N$

$\Rightarrow n_r > N$

$\Rightarrow |x_{n_r} - l| < \varepsilon$ by def

Hence proved $r > R \Rightarrow |x_{n_r} - l| < \varepsilon \quad \forall \varepsilon > 0$,

i.e. $\lim_{r\to\infty} x_{n_r} = l$ // QED.

Theorem 5.9

Every sequence has (at least one) monotonic subsequence.

Def x_N is called a "peak pt" if for the seq $\langle x_n \rangle$ $\forall n > N \Rightarrow x_n < x_N$.

Proof Case 1 There is infinite number of peak pts. $\Longleftrightarrow$ "restatement"

i.e. $N_1 < N_2 < N_3 < \cdots < N_r < \cdots$ $(\longrightarrow \infty)$ $\left\{ \forall N_r \ \exists N_{r+1} \text{ s.t. } N_{r+1} > N_r \Rightarrow x_{N_{r+1}} < x_{N_r} \right.$

corresponds to: $x_{N_1} > x_{N_2} > x_{N_3} > \cdots > x_{N_r} > x_{N_{r+1}} > \cdots$

we have a decreasing subsequence

Case 2 There is only a finite number of peak pts.

say $k_1, k_2, \cdots, k_m$ are the subscripts of the finite number of peak pts.

then let $n_1 = k_m + 1$,

then x_{n_1} is NOT a peak pt! i.e. $\exists n_2$ s.t. $n_2 > n_1 \Rightarrow x_{n_2} > x_{n_1}$

and x_{n_2} is NOT a peak pt $\because$ x_{k_m} is the LAST peak pt in the seq.

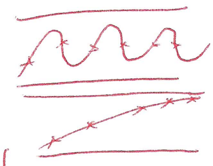
So i.e. $\forall n_r \ \exists n_{r+1}$ s.t. $x_{n_{r+1}} > x_{n_r}$

We have $x_{n_1} < x_{n_2} < \cdots < x_{n_r} < x_{n_{r+1}} < \cdots \longrightarrow \infty$

We have a strictly increasing subsequence. // QED.

Theorem – BOLZANO–WEIERSTRASS THEOREM

Every bounded sequence of $\mathbb{R}$ has a convergent subsequent.



Proof Let $\langle x_n \rangle_{n=1,2,\ldots}$ be a bounded sequence, i.e. $m \le x_n \le M$ $\forall n \in \mathbb{N}$.

By theorem 5.9 above, $\exists$ at least one monotone subsequence, call it $\langle x_{n_r} \rangle_{r=1,2,\ldots}$

Case 1 $\langle x_{n_r} \rangle$ is increasing,

But ~~all~~ $\forall r \in \mathbb{N}$, $n_r \in \mathbb{N}$, $x_{n_r} \le M$,

i.e. $\langle x_{n_r} \rangle$ is increasing & bounded above, $\underset{\text{theorem}}{\Rightarrow}$ $\langle x_{n_r} \rangle$ is convergent to its sup.

Case 2 $\langle x_{n_r} \rangle$ is decreasing,

Similarly, x_{n_r} is decreasing & bounded below by m,

theorem $\Rightarrow$ it is convergent to its inf.

Cauchy Sequence

Def A seq $\langle x_n \rangle$ is called Cauchy when: $\forall \varepsilon > 0 \ \exists N$ s.t. $m, n > N \Rightarrow |x_n - x_m| < \varepsilon$

General Principle of Convergence

A seq $\langle x_n \rangle$ is convergent iff $\langle x_n \rangle$ is Cauchy.

Proof A – Forward.

Given x_n is convergent, i.e. $\exists \ell$ s.t. $\lim x_n = \ell$

i.e. $\forall \varepsilon > 0 \ \exists N$ s.t. $n > N \Rightarrow |x_n - \ell| < \varepsilon$

then $\Rightarrow$ given $m > N$ we have $|x_m - \ell| < \varepsilon$ also, $\forall \varepsilon > 0$.

Consider when $\frac{\varepsilon}{2} > 0$, we have $\left(n > N \Rightarrow |x_n - \ell| < \frac{\varepsilon}{2}\right) \wedge \left(m > N \Rightarrow |x_m - \ell| < \frac{\varepsilon}{2}\right)$ true.

Adding gives: $|x_n - \ell| + |x_m - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

But $|x_n - \ell| - |x_m - \ell| = |x_n - \ell| + |\ell - x_m| \underset{\text{triangle}}{\geq} |x_n - \ell + \ell - x_m| = |x_n - x_m|$

Transitivity gives $|x_n - x_m| < \varepsilon \quad \forall n, m > N$, given $\varepsilon > 0$. // QED.

Proof B – Backwards

Lemma I $\langle x_n \rangle$ is Cauchy $\Rightarrow$ $\langle x_n \rangle$ is bounded.

Proof (I) $\langle x_n \rangle$ is Cauchy means $\forall \varepsilon > 0 \ \exists N$ s.t. $n, m > N \Rightarrow |x_n - x_m| < \varepsilon$

Consider $\varepsilon = 1$, then $n, m > N \Rightarrow |x_n - x_m| < 1 \Leftrightarrow (x_m - 1 < x_n < x_m + 1 \quad \forall n, m > N.)$

Consider $m = N+1 > N$, then $\forall n > N \Rightarrow x_{N+1} - 1 < x_n < x_{N+1} + 1$ ——— (1)

Define $M = \max(x_1, x_2, \dots, x_N, x_{N+1}+1)$ & $M' = \min(x_1, x_2, \dots, x_N, x_{N+1} - 1)$

Case 1: For $n \leq N$, $x_n \in (x_1, x_2, \dots, x_N)$

Hence $M' \leq x_n \leq M$

Case 2: For $n > N$, (1) gives $x_{N+1} - 1 < x_n < x_{N+1} + 1 \leq M$

Hence $M' < x_n < M$

Therefore $\langle x_n \rangle$ is bounded //

\# Bolzano-Weierstrass: If $\langle x_n \rangle$ is bounded, it has a convergent subsequence.

Lemma II If $\left\{ \begin{array}{l} \langle x_n \rangle \text{ is Cauchy} \\ \langle x_{n_r} \rangle \text{ is a subseq of } \langle x_n \rangle \text{ with } \lim_{r \to \infty} x_{n_r} = \ell \end{array} \right\}$ then $\langle x_n \rangle$ is convergent & converge to ℓ.

Proof (II)

Given $\left\{ \begin{array}{l} \forall \varepsilon > 0 \ \exists N \text{ s.t. } n, m > N \Rightarrow |x_n - x_m| < \varepsilon, \\ \forall \varepsilon > 0 \ \exists R \text{ s.t. } r > R \Rightarrow |x_{n_r} - \ell| < \varepsilon. \end{array} \right.$, Consider $\frac{\varepsilon}{2} > 0$:

Define $N_1 = \max(N, R)$, we have:

$\forall r > N_1 \Rightarrow r > R \Rightarrow |x_{n_r} - \ell| < \frac{\varepsilon}{2}$

Consider $m = n_r \underset{\text{lemma}}{\geq} r > N_1 \geq N \Rightarrow |x_n - x_{n_r}| < \frac{\varepsilon}{2}$

Now consider $|x_n - \ell|$ for only $n > N_1$

$= |x_n - x_{n_r} + x_{n_r} - \ell|$

$\leq |x_n - x_{n_r}| + |x_{n_r} - \ell|$ triangle

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ $\because n > N_1$

$= \varepsilon \quad \forall \varepsilon > 0$. $\therefore \forall \varepsilon > 0 \ \exists N_1$ s.t. $n > N \Rightarrow |x_n - \ell| < \varepsilon$

$\Rightarrow \lim_{n \to \infty} x_n = \ell$ // QED. | General Principle Proved // |

Series

Def The sequence of partial sum $\langle S_N \rangle$ of the series $\sum_{n=1}^{\infty} a_n$ is defined by $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_N$

If $\lim_{N\to\infty} S_N = s$, the series is said to converge to sum "s": $\sum_{n=1}^{\infty} a_n = s$.

4 Main types

① Geometric series $\sum_{n=1}^{\infty} x^n$

② Telescoping series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

③ Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

④ Riemann-Zeta function $\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, $\alpha > 1$

Theorem 6.5 The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$.

Proof - Partial sum increases $\because$ all pos terms $\quad S_N = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N}$

- Need to show unbounded above

$$
\begin{aligned}
S_{(2^N)} &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^N} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{N-1}+1} + \frac{1}{2^{N-1}+2} + \dots + \frac{1}{2^N}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^N} + \frac{1}{2^N} + \dots + \frac{1}{2^N}\right) \\
&= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{N} = 1 + N\left(\tfrac{1}{2}\right) \xrightarrow[N\to\infty]{} +\infty
\end{aligned}
$$

Theorem 6.6 $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges for $\alpha > 1$.

Proof - Partial sum increases, need to show bounded.

$$
\begin{aligned}
\underset{\text{For } N>1 \text{ onwards}}{S_N \le S_{(2^N-1)}} &= \underset{2^0}{1} + \left(\underset{2^1}{\frac{1}{2^\alpha}} + \underset{2^2-1}{\frac{1}{3^\alpha}}\right) + \left(\underset{2^2}{\frac{1}{4^\alpha}} + \frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \underset{2^3-1}{\frac{1}{7^\alpha}}\right) + \left(\underset{2^3}{\frac{1}{8^\alpha}} + \dots + \underset{2^4-1}{\frac{1}{15^\alpha}}\right) + \dots + \left(\underset{2^{N-1}}{\dots} + \frac{1}{(2^N-2)^\alpha} + \underset{2^N-1}{\frac{1}{(2^N-1)^\alpha}}\right) \\
&\le 1 + \left(\frac{1}{2^\alpha} + \frac{1}{2^\alpha}\right) + \left(\frac{1}{4^\alpha} + \frac{1}{4^\alpha} + \frac{1}{4^\alpha} + \frac{1}{4^\alpha}\right) + \dots + \left(\frac{1}{(2^{N-1})^\alpha} + \dots + \frac{1}{(2^{N-1})^\alpha}\right) \\
&= 1 + \frac{1}{2^\alpha} \times 2 + \frac{1}{4^\alpha} \times 4 + \dots + \frac{1}{(2^{N-1})^\alpha} \times 2^{N-1} \\
&= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{2^{2(\alpha-1)}} + \dots + \frac{1}{2^{(N-1)(\alpha-1)}} \\
&= 1 + \frac{1}{2^{\alpha-1}} + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \dots + \left(\frac{1}{2^{(\alpha-1)}}\right)^{N-1} \\
&= \frac{1\left(1 - \left(\frac{1}{2^{\alpha-1}}\right)^N\right)}{1 - \frac{1}{2^{\alpha-1}}} < \frac{1}{1 - \frac{1}{2^{\alpha-1}}} = \text{constant}
\end{aligned}
$$

i.e. $\langle S_N \rangle$ is bounded above, $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges.

Elementary properties 6.8

Given $\sum_{n=1}^{\infty} a_n$ converges & $\sum_{n=1}^{\infty} b_n$ converges,

(i) $\sum_{n=1}^{\infty}(a_n+b_n)$ converges and $\sum(a_n+b_n) = \sum a_n + \sum b_n$ — Associative law for series

(ii) Fix $\lambda \in \mathbb{R}$, $\sum \lambda a_n$ converges and $\sum(\lambda a_n) = \lambda \sum a_n$ — Distributive law for series

Remarks: it is NOT true $\sum(a_n b_n)$ converges ~~NOR~~ AND $\sum(a_n b_n) \neq (\sum a_n)(\sum b_n)$

Proof (i) Let S_N be the partial sums of $\sum_{n=1}^{\infty} a_n$ i.e. $S_N = a_1 + a_2 + \cdots + a_N$

Let T_N be the partial sums of $\sum_{n=1}^{\infty} b_n$ i.e. $T_N = b_1 + b_2 + \cdots + b_N$

Note: $\lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} a_n$ & $\lim_{N\to\infty} T_N = \sum_{n=1}^{\infty} b_n$;

Consider U_N as partial sum of $\sum_{n=1}^{\infty}(a_n+b_n)$,

i.e. $U_N = (a_1+b_1) + (a_2+b_2) + \cdots + (a_N+b_N)$

$= (a_1+a_2+\cdots+a_N) + (b_1+b_2+\cdots+b_N)$ associative law for addition

$= S_N + T_N$

$\lim_{N\to\infty} U_N = \lim_{N\to\infty}(S_N+T_N) = \lim_{N\to\infty} S_N + \lim_{N\to\infty} T_N$ combination for limits

$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ //

Proof (ii) Let the partial sum of $\sum_{n=1}^{\infty}(\lambda a_n)$ be $V_N = \lambda a_1 + \lambda a_2 + \cdots + \lambda a_N$

$= \lambda(a_1 + a_2 + \cdots + a_N)$

$= \lambda S_N$

$\lim_{N\to\infty} V_N = \lim_{N\to\infty} \lambda S_N = \lambda \lim_{N\to\infty} S_N$

$\sum_{n=1}^{\infty}(\lambda a_n) = \lambda \sum_{n=1}^{\infty} a_n$ //

CAUCHY'S CRITERION FOR CONVERGENCE OF SERIES

A series converges iff its partial sum is Cauchy.

i.e. $\sum_{n=1}^{\infty} a_n$ converges $\underset{\text{def}}{\iff}$ $\langle S_N \rangle$ converges $\underset{5.19}{\iff}$ $\langle S_N \rangle$ is Cauchy.

i.e. $\forall \varepsilon > 0\ \exists N$ s.t. $n > m > N \Rightarrow |S_n - S_m| < \varepsilon$

Here $S_n - S_m = (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_m)$

$= a_{m+1} + a_{m+2} + \cdots + a_n$

$= \sum_{i=m+1}^{n} a_i$

Hence $\langle S_N \rangle$ is Cauchy $\Rightarrow$ $\forall \varepsilon > 0\ \exists N$ s.t. $n > m > N \Rightarrow \left|\sum_{i=m+1}^{n} a_i\right| < \varepsilon$

Alternating Series Test $\left\{ \sum_{n=1}^{\infty} (-1)^{n-1} a_n \right\}$

Given a_n positive (non-negative) ; decreasing ; $\lim_{n\to\infty} a_n = 0$,
then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges .

Proof Let S_N be the partial sums ; $S_N = a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$

$S_1 = a_1$ $\qquad a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots$

$S_2 = a_1 - a_2$ $\qquad a_2 - a_3 \geq 0 \;/\; a_3 - a_4 \geq 0$

$S_3 = a_1 - a_2 + a_3 = a_1 - (a_2 - a_3) \leq a_1 = S_1$

$S_4 = a_1 - a_2 + (a_3 - a_4) \geq a_1 - a_2 = S_2$

Prop S_{2n-1} is decreasing ; S_{2n} is increasing ; $S_{2n-1} \geq S_{2n}$

$S_{2n+1} = S_{2n-1} - a_{2n} + a_{2n+1}$
$\quad = S_{2n-1} - (a_{2n} - a_{2n+1})$
$S_{2n+1} \leq S_{2n-1}$

$a_{2n+1} \leq a_{2n}$
$a_{2n} - a_{2n+1} \geq 0$

$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2})$
$S_{2n+2} \geq S_{2n}$

$a_{2n+2} \leq a_{2n+1}$
$a_{2n+1} - a_{2n+2} \geq 0$

$S_{2n} = S_{2n-1} - a_{2n} \leq S_{2n-1} \Rightarrow S_{2n} \leq S_{2n-1} \quad \forall n \in \mathbb{N}$

Now S_{2n-1} is decreasing, hence $S_1 \geq S_3 \geq \cdots$

let $S_1 = b$, then $b \geq S_{2n-1} \quad \forall n \in \mathbb{N} \Rightarrow$

by transitivity $b \geq S_{2n} \quad \forall n \in \mathbb{N}$

then S_{2n} is increasing & bounded above $\Rightarrow$ a $\lim_{n\to\infty} S_{2n}$ exists.

Similarly S_{2n-1} is decreasing & bounded below $\Rightarrow \lim_{n\to\infty} S_{2n-1}$ exists.

$S_{2n} = S_{2n-1} - a_{2n}$

$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty}(S_{2n-1} - a_{2n}) = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} a_{2n}$, But $\lim_{n\to\infty} a_n = 0$

$\therefore \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n-1}$

$\Rightarrow$ The Alternating Series converges .

N-th term test for divergence 6.9

(a) If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$ $\Leftrightarrow$

(b) If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof $\sum_{n=1}^{\infty} a_n$ converge means $\lim_{N\to\infty} S_N = s$ where s is a constant and $S_N = \sum_{n=1}^{N} a_n$

$S_N = S_{N-1} + a_N \Leftrightarrow a_N = S_N - S_{N-1}$

$\lim_{N\to\infty} a_N = \lim_{N\to\infty}(S_N - S_{N-1}) \overset{\text{comb.}}{=} \lim_{N\to\infty} S_N - \lim_{N\to\infty} S_{N-1} = s - s = 0$ // QED.

Comparison Test

Given $|a_n| \le b_n$;

(i) If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges.

Proof (i) $\sum_{n=1}^{\infty} b_n$ converges $\underset{\text{Cauchy Criterion}}{\Leftrightarrow}$ $\forall \varepsilon>0\ \exists N$ s.t. $n>m>N \Rightarrow \left|\sum_{i=m+1}^{n} b_i\right| < \varepsilon$, $x \le |x|$

$\Rightarrow \sum_{i=m+1}^{n} b_i < \varepsilon$

$\Rightarrow \sum_{i=m+1}^{n} |a_i| < \varepsilon$, $|a+b| \le |a|+|b|$

$\Rightarrow \left|\sum_{i=m+1}^{n} a_i\right| < \varepsilon$

$\Rightarrow \left(\forall \varepsilon>0\ \exists N \text{ s.t. } n>m>N \Rightarrow \left|\sum_{i=m+1}^{n} a_i\right| < \varepsilon\right)$

$\Leftrightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

Ratio Test

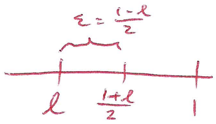
Given $\sum_{n=1}^{\infty} a_n$,

(i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow$ convergent

(ii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow$ divergent

(iii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow$ inconclusive

Proof (i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l < 1 \Rightarrow \left(\forall \varepsilon>0\ \exists N \text{ s.t. } n>N \Rightarrow \left|\left|\frac{a_{n+1}}{a_n}\right| - l\right| < \varepsilon\right)$



Take $\varepsilon = \frac{1-l}{2}$, let $N \in \mathbb{N}$ any;

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| - l < \varepsilon$ $[x \le |x|]$

$\Rightarrow \frac{|a_{n+1}|}{|a_n|} < \varepsilon + l$

$\Rightarrow |a_{n+1}| < (l+\varepsilon)|a_n|$

we have:

$$\left.\begin{array}{ll} \text{for } n=N+1 & |a_{N+2}| < (l+\varepsilon)|a_{N+1}| \\ n=N+2 & |a_{N+3}| < (l+\varepsilon)|a_{N+2}| \\ \vdots & \vdots \\ n=n-1 & |a_n| < (l+\varepsilon)|a_{n-1}| \end{array}\right\} \times \text{ times}$$

$$\Rightarrow |a_{N+2}|\cdot|a_{N+3}|\cdots|a_n| < (l+\varepsilon)^{(n-1)-(N+1)+1}\cdot|a_{N+1}|\cdot|a_{N+2}|\cdots|a_{n-1}|$$

cancel

$$|a_n| < (l+\varepsilon)^{n-1-N}|a_{N+1}| = (l+\varepsilon)^n \underbrace{\left(|a_{N+1}|\cdot(l+\varepsilon)^{-N-1}\right)}_{\text{constant}}$$

$$\sum_{n=1}^{\infty}|a_n| < |a_{N+1}|\cdot(l+\varepsilon)^{-N-1}\sum_{n=1}^{\infty}(l+\varepsilon)^n \leftarrow \text{GP, ratio} = l+\varepsilon = \frac{1+l}{2} < 1 \because l<1$$

$\Rightarrow$ RHS converge.

By comparison test LHS converge;

Hence $\sum_{n=1}^{\infty} a_n$ converges absolutely. (6.20) shown later.

Root Test

Compute $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l$

(i) $l < 1 \Rightarrow \sum a_n$ converges

(ii) $l > 1 \Rightarrow \sum a_n$ diverges

(iii) $l = 1 \Rightarrow \sum a_n$ inconclusive.

Theorem of Abs convergence 6.20

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof $a_n \leq |a_n| \; \forall n \in \mathbb{N}$.

obvious by comparison test $\sum a_n$ converges if $\sum |a_n|$ converge.

Corollary If $\sum a_n$ diverges, $\sum |a_n|$ diverges.

Def - If $\sum |a_n|$ converges, we say $\sum a_n$ is "absolutely convergent".

- If $\sum |a_n|$ diverges while $\sum a_n$ converges, we say $\sum a_n$ is "conditionally convergent."

$\boxed{\text{SUMMARY}}$

[1] N-th term test for divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

[2] Comparison test

Given $|a_n| \leq b_n$, If $\sum b_n$ converges, $\sum a_n$ converges

[3] Ratio test

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l$, if $l < 1$, $\sum a_n$ converges;
$l > 1$, $\sum a_n$ diverges;
$l = 1$, $\sum a_n$ is inconclusive.

[4] Root test Consider $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l$, conclusions identical to ratio test.

[5] Alternating series test Given $\sum (-1)^n a_n$, $\lim_{n\to\infty} a_n = 0 \Rightarrow$ series converges.

[6] Theorem of Abs convergence If $\sum |a_n|$ converges, $\sum a_n$ converges (absolutely).

Theorem for rearrangements of series

(a) If $\sum a_n$ converges absolutely, then any rearrangements $\sum a_{f(n)}$ converges to same sum.
i.e. $\sum a_{f(n)} = \sum a_n$

(b) If $\sum a_n$ converges conditionally, Fix a real number "s", $\exists$ a rearrangement s.t. $\sum a_n = s$.

LIMIT COMPARISON TEST

Theorem let $a_n > 0$ & $b_n > 0$;

(i) $\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, \infty)$ where c is a constant,

BOTH Σa_n & Σb_n conerge or dierge together.

(ii) $\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \Rightarrow$ If Σb_n converges, Σa_n conerges.

(iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \Rightarrow$ If Σb_n dierges, Σa_n dierges.

Proof let $\lim_{n\to\infty} \frac{a_n}{b_n} = l \Rightarrow \left(\forall \varepsilon > 0, \exists N \text{ s.t. } n > N \Rightarrow \left|\frac{a_n}{b_n} - l\right| < \varepsilon\right)$

Take $\varepsilon = \frac{l}{2} > 0$, $\exists N$ s.t. $n > N \Rightarrow \left|\frac{a_n}{b_n} - l\right| < \frac{l}{2}$

$$\Leftrightarrow \quad -\frac{l}{2} < \frac{a_n}{b_n} - l < \frac{l}{2}$$

$$\Leftrightarrow \quad \frac{l}{2} < \frac{a_n}{b_n} < \frac{3l}{2}$$

$$\Leftrightarrow \quad \frac{l}{2} \cdot b_n < a_n < \frac{3l}{2} b_n$$

(i) If Σb_n conerge $\Rightarrow \Sigma \frac{3l}{2} b_n$ conerges $\underset{\text{comparison}}{\Rightarrow} \Sigma a_n$ conerges //

Σa_n conerge $\underset{\text{comparison}}{\Rightarrow} \Sigma \frac{l}{2} b_n$ conerges $\Rightarrow \Sigma b_n$ conerges //

i.e. Σa_n ~~& Σb_n~~ conerges iff Σb_n conerges.

$\Rightarrow \Sigma a_n$ dierges iff Σb_n dierges.

— END OF SERIES —

FUNCTIONS — Limits of functions

Def $\lim_{x \to b^-} f(x) = l$ means $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $b - \delta < x < b \Rightarrow |f(x) - l| <$

Def $\lim_{x \to b^+} f(x) = l$ means $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $b < x < b + \delta \Rightarrow |f(x) - l| < \varepsilon$

Def $\lim_{x \to \xi} f(x) = l$ means $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x - b| < \delta \Rightarrow |f(x) - l| < \varepsilon$

Proposition 8.4

let $f : (a,b) \to \mathbb{R}$ and $\xi \in (a,b)$, " $\lim_{x \to \xi} f(x) = l$ " iff " $\lim_{x \to \xi^+} f(x) = \lim_{x \to \xi^-} f(x) = l$ "

Example 1 $\lim_{x \to 2^+} f(x) = 4$ for $f(x) = x^2$. Prove.

$\lim_{x \to 2^+} f(x) = 4 \Rightarrow \left(\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 2 < x < 2 + \delta \Rightarrow |f(x) - 4| < \varepsilon \right)$

$|f(x) - 4| < \varepsilon$

$\Leftrightarrow -\varepsilon < x^2 - 4 < \varepsilon$

$\Leftrightarrow 4 - \varepsilon < x^2 < \cancel{\varepsilon + 4}\, 4 + \varepsilon$

$$\left[\begin{aligned} & 2 < x < 2 + \delta \\ \Leftrightarrow\ & 4 < x^2 < (2+\delta)^2 \\ \Rightarrow\ & 4 - \varepsilon < x^2 < (2+\delta)^2 \\ \Rightarrow\ & 4 - \varepsilon < x^2 < 4 + \varepsilon \\ \Rightarrow\ & |f(x) - 4| < \varepsilon \quad // \end{aligned}\right]$$

let ~~Take~~ $(2+\delta)^2 = 4 + \varepsilon$

$2 + \delta = \sqrt{4 + \varepsilon}$

Take: $\delta = \sqrt{4 + \varepsilon} - 2$

$\Rightarrow (2+\delta)^2 = 4 + \varepsilon$

Also, given $\varepsilon > 0$

$4 + \varepsilon > 4 \ (> 0)$

$\sqrt{4 + \varepsilon} > 2$

$\sqrt{4 + \varepsilon} - 2 > 0 \Rightarrow \delta > 0$

$\therefore$ Take $\delta = \sqrt{4 + \varepsilon} - 2 > 0$, we have $2 < x < 2 + \delta \Rightarrow |f(x) - 4| < \varepsilon, \forall \varepsilon > 0$, i.e. $\lim_{x \to 2^+} f(x) = 4$ //

Example 2 $f(x) = x \sin\left(\frac{1}{x}\right)$, $x \neq 0$. Prove $\lim_{x \to 0} f(x) = 0$.

i.e. prove $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| < \varepsilon$

Take $\delta = \varepsilon > 0$: $|x - 0| < \delta$

$\Rightarrow |x| < \varepsilon$

But $|\sin(\frac{1}{x})| \leq 1$

$\Leftrightarrow |x||\sin(\frac{1}{x})| \leq |x|$

~~$|x| < |x|$~~

$\Rightarrow |x||\sin(\frac{1}{x})| \leq |x| < \varepsilon$

$\Rightarrow |x \sin(\frac{1}{x})| < \varepsilon$

$\Rightarrow |f(x) - 0| < \varepsilon$ // QED.

FUNCTION CONTINUITY (at a pt)

Def 8.4

$\boxed{1}$ let f be defined on (a,b) with $\xi \in (a,b)$. Then f is cts at ξ if

$\lim_{x\to\xi} f(x)$ exists (i.e. $\lim_{x\to\xi^+} f(x) = \lim_{x\to\xi^-} f(x)$) $\wedge$ $\lim_{x\to\xi} f(x) = f(\xi)$.

$\boxed{2}$ let f be defined on $(a,b]$ ~~with ξ~~ Then f is cts at b from the left if

$\lim_{x\to b^-} f(x)$ exists & $\lim_{x\to b^-} f(x) = f(b)$

$\boxed{3}$ let f be defined on $[a,b)$. Then f is cts at a from the right if

$\lim_{x\to a^+} f(x)$ exists & $\lim_{x\to a^+} f(x) = f(a)$.

$\boxed{4}$ If $f:[a,b]\to\mathbb{R}$, we say f is cts on the compact interval $[a,b]$ if:

(a) $\forall \xi \in (a,b)$, f is cts at ξ (by $\boxed{1}$)
(b) f is continuous from left at b (by $\boxed{2}$)
(c) f is continuous from right at a (by $\boxed{3}$)

All satisfy

Example All linear functions are cts. Prove!

let $f(x) = Ax + B$.

Case 1 $A = 0$: $f(x) = B$ $\lim_{x\to\xi} f(x) = \lim_{x\to\xi} B = B = f(\xi)$

Case 2 $A \neq 0$: To prove $\lim_{x\to\xi} f(x) = f(\xi)$ $\forall \xi \in (-\infty, \infty)$

$\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $(|x-\xi| < \delta \Rightarrow |f(x) - f(\xi)| < \varepsilon)$

$|f(x) - f(\xi)| < \varepsilon$
$\Leftrightarrow |Ax + B - (A\xi + B)| < \varepsilon$
$\Leftrightarrow |A||x-\xi| < \varepsilon$
$\Leftrightarrow |x - \xi| < \frac{\varepsilon}{|A|}$

Take $\delta = \frac{\varepsilon}{|A|} > 0$;

Hence $|x-\xi| < \delta \Rightarrow |f(x) - f(\xi)| < \varepsilon$

$\therefore$ prop proved. QED

rem 8.9

$\lim_{x\to\xi} f(x) = l$ iff $\forall$ sequence $\langle x_n \rangle$ with $x_n \neq \xi$ we have

$\lim_{n\to\infty} x_n = \xi \Rightarrow \lim_{n\to\infty} f(x_n) = l$ for $a \in \mathbb{R}$

Proof - Forward

Assume $\lim_{x\to\xi} f(x) = l$ true ; i.e. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $0 < |x - \xi| < \delta \Rightarrow |f(x) - l| < \varepsilon$

Then proof implication true:

$\lim_{n\to\infty} x_n = \xi \Rightarrow [\forall \varepsilon > 0\ \exists N \text{ s.t. } n > N \Rightarrow |x_n - \xi| < \varepsilon]$

$\Rightarrow [\forall \delta > 0\ \exists N \text{ s.t. } n > N \Rightarrow |x_n - \xi| < \delta]$ change of variable

$\because x_n \neq \xi,\ |x_n - \xi| > 0$

i.e. $n > N \Rightarrow 0 < |x_n - \xi| < \delta$

$\Rightarrow |f(x_n) - l| < \varepsilon$ *from assumption

Hence $[\forall \varepsilon > 0\ \exists N \text{ s.t. } n > N \Rightarrow |f(x_n) - l| < \varepsilon]$

$\Rightarrow \lim_{n\to\infty} f(x_n) = l$ //.

Proof - Backwards.

Converse of theorem: Given "$\lim_{n\to\infty} \langle x_n \rangle = \xi \Rightarrow \lim_{n\to\infty} f(x_n) = l$ $\forall$ seq $\langle x_n \rangle$ with $x_n \neq \xi$";

then we have "$\underline{\lim_{x\to\xi} f(x) = l}$". ← Aim to prove.

Assumption: Assume Given $\lim_{n\to\infty} x_n = \xi \Rightarrow \lim_{n\to\infty} f(x_n) = l$, I have $\lim_{x\to\xi} f(x) \neq l$.

$\lim_{x\to\xi} f(x) \neq l$ means $\exists \varepsilon > 0$ s.t. $\forall \delta > 0, \exists x$ s.t. $\underbrace{(0 < |x - \xi| < \delta)}_{A} \wedge \underbrace{(|f(x) - l| \geq \varepsilon)}_{B}$

$\Rightarrow A \wedge B$ Both True $\forall \delta > 0$

Consider $\delta = \frac{1}{n} > 0$; Then I take $x(\delta) = x_n$,

$0 < |x_n - \xi| < \frac{1}{n}$ $\boxed{\text{true}}$ $\wedge$ $|f(x_n) - l| \geq \varepsilon$ $\boxed{\text{true}}$

$\Leftrightarrow \xi - \frac{1}{n} < x_n < \xi + \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty}(\xi - \frac{1}{n}) \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty}(\xi + \frac{1}{n})$

$\Rightarrow \lim_{n\to\infty} x_n = \xi$ Sandwich theorem.

$\Rightarrow \lim_{n\to\infty} f(x_n) = l$ $\left\{ \boxed{\begin{array}{l}\text{Given } \lim_{n\to\infty} x_n = \xi \Rightarrow \lim_{n\to\infty} f(x_n) = l \\ \text{if } x \neq \xi \text{, here } |x_n - \xi| > 0 \Rightarrow x_n \neq \xi\end{array}} \right.$

$\Rightarrow \forall \varepsilon' > 0\ \exists N$ s.t. $n > N \Rightarrow |f(x_n) - l| < \varepsilon'$

contradicts !

$\therefore$ Assumption of $\lim_{x\to\xi} f(x) \neq l$ fails when $\delta = \frac{1}{n} > 0$;

Hence $\neg(\lim_{x\to\xi} f(x) \neq l)$ is true ; $\therefore \lim_{x\to\xi} f(x) = l$ given above conditions //

QED.

Theorem 8.9 simplifies the work to prove limit of $f(x)$'s.

Example $\lim_{x\to 5} x^2 = 25$. Prove! where $f(x)=x^2$

Theorem 8.9 says $\forall$ seq $\langle x_n\rangle$ with $\lim_{n\to\infty} x_n = \xi \Rightarrow \lim_{n\to\infty} f(x_n) = l$ where $x_n \neq \xi$, then $\lim_{x\to\xi} f(x) = l$

let $\xi = 5$, take any seq $\langle x_n\rangle$ where $x_n \neq 5$ but $\lim_{n\to\infty} x_n = 5$;

Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n)^2 = \lim_{n\to\infty} x_n \cdot x_n \underset{\text{combination of eseq.}}{=} \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n = 5\cdot 5 = 25 = l$

Hence $\lim_{x\to\xi} f(x) = 25$ or $\lim_{x\to 5} x^2 = 25$ // QED.

Proposition: All polynomials are continuous with degree n for $n\in\mathbb{N}$

Proof let $P(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x^1 + a_0$

Aim to prove: $\lim_{x\to\xi} P(x) = P(\xi) \quad \forall \xi \in \mathbb{R}$.

Construct sequence $\langle x_n\rangle$ where $x_n \neq \xi$ with $\lim_{n\to\infty} x_n = \xi$ for $n\in\mathbb{N}$.

$$\text{Then } \lim_{n\to\infty} P(x_n) = \lim_{n\to\infty}\left[a_k x_n^k + a_{k-1}x_n^{k-1} + \cdots + a_1 x_n^1 + a_0\right]$$

$$\overset{\text{comb (a)}}{=} \left(\lim_{n\to\infty} a_k x_n^k\right) + \left(\lim_{n\to\infty} a_{k-1}x_n^{k-1}\right) + \cdots + \left(\lim_{n\to\infty} a_1 x_n^1\right) + \left(\lim_{n\to\infty} a_0\right)$$

$$\overset{\text{comb (b)}}{=} a_k\left(\lim_{n\to\infty} x_n^k\right) + a_{k-1}\left(\lim_{n\to\infty} x_n^{k-1}\right) + \cdots + a_1\left(\lim_{n\to\infty} x_n\right) + a_0$$

$$\overset{\text{comb (c)}}{=} a_k\left(\lim_{n\to\infty} x_n\right)^k + a_{k-1}\left(\lim_{n\to\infty} x_n\right)^{k-1} + \cdots + a_1\left(\lim_{n\to\infty} x_n\right) + a_0$$

$$\overset{\text{by construction}}{=} a_k \xi^k + a_{k-1}\xi^{k-1} + \cdots + a_1 \xi^1 + a_0$$

$$= P(\xi) = l$$

Given seq $\langle x_n\rangle$ $n\in\mathbb{N}$ where $x_n \neq \xi$ with $\lim_{n\to\infty} x_n = \xi \Rightarrow \lim_{n\to\infty} P(x_n) = P(\xi)$; $\forall \xi\in\mathbb{R}$

Theorem 8.9 gives $\lim_{x\to\xi} P(x) = P(\xi) \quad \forall \xi \in \mathbb{R}$.

By def $P(x)$ is cts at ξ $\forall \xi \in \mathbb{R}$. // QED.

$\Rightarrow$ Given $\lim_{x\to\xi} f(x) = l$ & $\lim_{x\to\xi} g(x) = m$;

(a) $\lim_{x\to\xi} (f(x)+g(x)) = l+m$

(b) $\lim_{x\to\xi} (\lambda f(x)) = \lambda \cdot l$ given $\lambda \in \mathbb{R}$

(c) $\lim_{x\to\xi} (f(x) \cdot g(x)) = l \cdot m$

(d) $\lim_{x\to\xi} \left(\frac{f(x)}{g(x)}\right) = \frac{l}{m}$ for $g(x) \neq 0$ & $m \neq 0$

Proof (a)

Take $\langle x_n \rangle$ where $n = 1, 2, \cdots$, $x_n \neq \xi$ and $\lim_{n\to\infty} x_n = \xi$;

$\because$ Already given $\lim_{x\to\xi} f(x) = l$ & $\lim_{x\to\xi} g(x) = m$, 8.9 tells us that

the construction of $\langle x_n \rangle$ with $\lim_{n\to\infty} x_n = \xi \Rightarrow \lim_{n\to\infty} f(x_n) = l$ & $\lim_{n\to\infty} g(x_n) = m$.

Consider $\lim_{n\to\infty} (f(x_n) + g(x_n)) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n)$ by combination of seqs.

$= l + m$

By 8.9 : $\lim_{x\to\xi} (f(x)+g(x)) = l+m$ // QED

Proof (c) Similar, replace all $(+)$ to $(\times)$.

Proof (b) Similar with (c), take $g(x) = \lambda$ as constant

Proof (d) let $m \neq 0$, Take $\langle x_n \rangle$ $n = 1, 2, \ldots$ where $x \neq \xi$ and $\lim_{n\to\infty} x_n = \xi$,

8.9 $\Rightarrow$ I have $\lim_{n\to\infty} f(x_n) = l$ & $\lim_{n\to\infty} g(x_n) = m$ $\because$ given $\lim_{x\to\xi} f(x) = l$ and $\lim_{x\to\xi} g(x) = m$.

Consider $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{l}{m}$ by combination of sequences

already given seq $\langle x_n \rangle$ with $x \neq \xi$ & $\lim_{n\to\infty} x_n = \xi$; 8.9 gives $\lim_{x\to\xi} \frac{f(x)}{g(x)} = \frac{l}{m}$ // QED.

\# Consequence : All rational functions are continuous for $\xi \neq$ root of $Q(x)$.

Proof let $\frac{P(x)}{Q(x)}$ be the quotient of 2 polynomials.

Assume $Q(x) \neq 0$, then

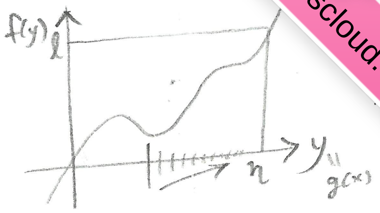
$$\lim_{x\to\xi} \frac{P(x)}{Q(x)} \overset{(d)}{=} \frac{\lim_{x\to\xi} P(x)}{\lim_{x\to\xi} Q(x)} \overset{\text{all poly cts}}{=} \frac{P(\xi)}{Q(\xi)}$$

; define $f(x) = \frac{P(x)}{Q(x)}$, then $\lim_{x\to\xi} f(x) = f(\xi)$.

Hence all rational func are cts. // QED

Theorem 8.17

Given $\lim_{y\to\eta} f(y) = l$ & $\lim_{x\to\xi} g(x) = \eta$, then $\lim_{x\to\xi} f(g(x)) = l$



if either of the following is true:

(i) $g(x) \neq \eta$ for x in an interval around ξ, or

(ii) f is cts at η

Proof

Given $\lim_{y\to\eta} f(y) = l \Rightarrow \forall \varepsilon > 0 \ \exists \Delta > 0$ s.t. $0 < |y-\eta| < \Delta \Rightarrow |f(y) - l| < \varepsilon$ — ①

$\lim_{x\to\xi} g(x) = \eta \Rightarrow \forall \Delta > 0 \ \exists \delta > 0$ s.t. $0 < |x-\xi| < \delta \Rightarrow |g(x) - \eta| < \Delta$ — ②

Aim to prove $\lim_{x\to\xi} f(g(x)) = l$, i.e. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x-\xi| < \delta \Rightarrow |f(g(x)) - l| < \varepsilon$

let $y = g(x)$;

① becomes $\forall \varepsilon > 0, \exists \Delta > 0$ s.t. $\begin{cases} \exists \delta > 0 \text{ s.t. } 0 < |x-\xi| < \delta \Rightarrow |g(x) - \eta| < \Delta, \\ \text{and } \quad 0 < |y-\eta| < \Delta \Rightarrow |f(y) - l| < \varepsilon \end{cases}$

$0 < |g(x) - \eta| < \Delta \Rightarrow |f(g(x)) - l| < \varepsilon$

* given condition (i) $g(x) \neq \eta$, $\Rightarrow |g(x) - \eta| > 0$

then we have $0 < |x-\xi| < \delta \Rightarrow 0 < |g(x) - \eta| < \Delta \Rightarrow |f(g(x)) - l| < \varepsilon$

i.e. $\lim_{x\to\xi} f(g(x)) = l$.

* given condition (ii), $\lim_{y\to\eta} f(y) = f(\eta) = l$

① becomes $\forall \varepsilon > 0 \ \exists \Delta > 0$ s.t. $|y - \eta| < \Delta \Rightarrow |f(y) - l| < \varepsilon$

combine with ②:

$\forall \varepsilon > 0 \ \exists \Delta > 0$ s.t. $\begin{cases} \exists \delta > 0 \text{ s.t. } 0 < |x-\xi| < \delta \Rightarrow |g(x) - \eta| < \Delta \\ \quad |y - \eta| < \Delta \Rightarrow |f(y) - l| < \varepsilon \end{cases}$

$|g(x) - \eta| < \Delta \Rightarrow |f(g(x)) - l| < \varepsilon$

Hence we have:

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |x-\xi| < \delta \Rightarrow |g(x) - \eta| < \Delta \Rightarrow |f(g(x)) - l| < \varepsilon$

i.e. $\lim_{x\to\xi} (f(g(x))) = l$ // QED.

FUNCTION CONTINUITY (on an interval)

Def 9.1 (a) Suppose $f : (a,b) \to \mathbb{R}$, f is called cts on (a,b) if f is cts at ξ, $\forall \xi \in (a,b)$.

i.e. $\forall \xi \in (a,b)$: $\lim_{x \to \xi} f(x) = f(\xi)$

(b) Suppose $f : [a,b] \to \mathbb{R}$, f is called cts on $[a,b]$ if f is cts $\forall \xi \in (a,b)$ $\wedge$ right-cts at "a" $\wedge$ left-cts at "b"

i.e. (i) $\lim_{x \to \xi} f(x) = f(\xi)$ $\forall \xi \in (a,b)$,

(ii) $\lim_{x \to a^+} f(x) = f(a)$,

(iii) $\lim_{x \to b^-} f(x) = f(b)$,

Def The closed interval $[a,b]$, where $a, b \in \mathbb{R}$ is called "compact interval".

Prop 9.3 f is cts on I iff $\forall x \in I$, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $(y \in I \wedge |x-y| < \delta) \Rightarrow |f(x)-f(y)| < \varepsilon$

Proof one-step prove by def of limit of funct.

Prop 9.4 Combination Theorem for continuity of func on an interval.

Let $f, g : I \to \mathbb{R}$ cts on I; then

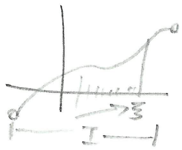
(a) $f+g$ is cts on I

(b) $\lambda \cdot f$ is cts on I

(c) $f \cdot g$ is cts on I

(d) $\frac{f}{g}$ is cts on I if $g(x) \neq 0$ on I.

Prop 9.5 Let $f : I \to \mathbb{R}$ and $g : J \to I$ & f, g are cts, then $f \circ g : J \to \mathbb{R}$ is cts.



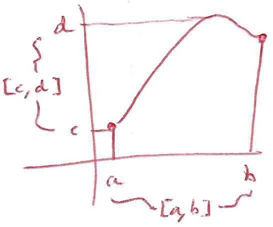
Prop 9.6 Let $f : I \to \mathbb{R}$ be cts.

Let $\langle x_n \rangle$ be a seq with $x_n \in I$ and $\lim_{n \to \infty} x_n = \xi$ where $\xi \in I$.

Then $\lim_{n \to \infty} f(x_n) = f(\xi) = f\left(\lim_{n \to \infty} x_n\right)$, i.e. the symbols commute.

CONTINUITY PROPERTY FOR CONTINUOUS FUNCTIONS

Theorem 9.8 Let $f: [a,b] \to \mathbb{R}$ be cts on the compact interval $[a,b]$. Then the image of $[a,b]$ under f is also compact interval.



Proof Split into 3 lesser theorem and prove seperately.

(i) Given f is cts on $[a,b] = I$, theorem 9.9 $\Rightarrow f(I) = J$
J is an interval.

(ii) 9.11 $\Rightarrow$ interval J is bounded

(iii) 9.12 $\Rightarrow$ J includes its endpts.

Theorem 9.9 Let f be cts on an interval I. Then the image of I under f is also an interval.

Corollary INTERMEDIATE VALUE THEOREM (Bolzano)
Let $f: [a,b] \to \mathbb{R}$ be cts on the compact interval $[a,b]$.
Let λ be a real number b/w $f(a)$ & $f(b)$. Then, there exists (at least) one real number $\xi \in (a,b)$ with $f(\xi) = \lambda$.
i.e. $\forall \lambda \in (f(a), f(b))$, $\exists \xi \in (a,b)$ s.t. $f(\xi) = \lambda$.

recall: A func f is called bounded on I if $\exists m, M$ s.t. $m \leq f(x) \leq M \quad \forall x \in I$.

Theorem 9.11 Let $f: [a,b] \to \mathbb{R}$ be cts on $[a,b]$, then f is bounded on $[a,b]$.

Theorem 9.12 Let $f: [a,b] \to \mathbb{R}$ be cts on $[a,b]$. Then f achieves maximum & minimum values on $[a,b]$.
i.e. $\exists c, d \in [a,b]$ s.t. $f(c) \leq f(x) \leq f(d) \quad \forall x \in [a,b]$

...plications of IVT

#1 Show that the following equation has a solution in the interval (...

$x^3+x-1=0$

Proof Let $f(x)=x^3+x-1$ and f is cts (polynomial) on $[0,1]$

~~Let $\lambda=0$~~. $f(0)=0+0-1=-1<0$, $f(1)=1+1-1=1>0$ } Take $\lambda=0$

we have $f(0)<0<f(1)$;

IVT $\Rightarrow \exists\, \xi\in(0,1)$ with $f(\xi)=0$. Hence solution for $f(x)$ exists.

#2 Let $f:[0,1]\to[0,1]$ be cts on $[0,1]$. Prove $\exists\,\xi\in[0,1]$ s.t. $f(\xi)=\xi$.

i.e. There exist at least one intersection b/w $y=f(x)$ & $y=x$.

Proof Let $g(x)=f(x)-x$ and g is cts on $[0,1]$

Case 1 ($g(0)\neq0 \wedge g(1)\neq0$, $g:[0,1]\to(0,1)$)

Consider $g(0)=f(0)-0=f(0)>0$, $g(1)=f(1)-1<0$ } $\because g:[0,1]\to(0,1)$

Take $\lambda=0$ we have $g(1)<\lambda<g(0)$

IVT $\Rightarrow \exists\,\xi\in(0,1)$ s.t. $g(\xi)=\lambda=0 \Rightarrow f(\xi)-\xi=0 \Rightarrow f(\xi)=\xi$.

Case 2 ($g(0)=0$)

$\Rightarrow f(0)-0=0$, $f(0)=0$

Take $\xi=0$ QED

Case 3 ($g(1)=0$)

$\Rightarrow f(1)-1=0$, $f(1)=1$ Take $\xi=1$ QED.

#3 Let $f:\mathbb{R}\to\mathbb{R}$ be cts for $f(x)$: $\dfrac{x\cdot f(x)+5}{1+f(x)^2}=\dfrac{1}{2}$, $x\in\mathbb{R}$, $f(0)=3$

Show that $f(x)=x+\sqrt{x^2+9}$, $\forall x\in\mathbb{R}$

Proof $\dfrac{x\cdot f(x)+5}{1+f(x)^2}=\dfrac{1}{2} \iff f(x)=x\pm\sqrt{x^2+9}$.

At $x=0$, ~~f(0)=3~~, $f(x)=x+\sqrt{x^2+9}$ s.t. $f(0)=3$.

Assume For $x=x_0$, $f(x)=x-\sqrt{x^2+9}$

Now $f(x_0)=x_0-\sqrt{x_0^2+9}<0$ $\quad\because \left\{ x_0^2+9>x_0^2,\ \sqrt{x_0^2+9}>x_0,\ x_0-\sqrt{x_0^2+9}<0 \right\}$

But $f(0)=3>0$

Take $\lambda=0$, ~~if $x_0<0$~~ then $f(x_0)<0<f(0)$

IVT $\Rightarrow \exists\,\xi$ ~~s.t.~~ $\in(0,x_0)$ s.t. $f(\xi)=0$

$\dfrac{x\cdot f(x)+5}{1+f(x)^2}=\dfrac{1}{2}$, at $x=\xi$:

$\dfrac{\xi\cdot f(\xi)+5}{1+f(\xi)^2}=\dfrac{1}{2}$

$\iff \dfrac{\xi\cdot 0+5}{1+0}=\dfrac{1}{2}$

$\iff 5=\dfrac{1}{2}$!! Contradiction.

~~$\therefore \nexists\,\xi\in(0,x_0)$ s.t. $f(\xi)$~~

$\therefore \nexists\, x_0\in\mathbb{R}$ s.t. $f(x)=x-\sqrt{x^2+9}$

i.e. $f(x)=x+\sqrt{x^2+9}$, $\forall x\in\mathbb{R}$.

$\sqrt{y}$ exists given $y \geq 0$

* Recall prove by continuum property of existence of $\sqrt{2}$.

Proof Let $f(x) = x^2$ is cts on $[0,\infty)$ where $f: [0,\infty) \to \mathbb{R}$.

Aim: $\exists \xi \geq 0$ s.t. $\xi^2 = y \iff \sqrt{y}$ exists.

Case 1 $y=0$, choose $\xi = 0 \Rightarrow 0^2 = 0 \Rightarrow \xi^2 = y$ satisfied i.e. $\sqrt{y}$ exists for $y=0$.

Case 2 $y=1$, choose $\xi = 1$: $1^2 = 1 \Rightarrow \xi^2 = y$ satisfied i.e. $\sqrt{y}$ exists for $y=1$

Case 3 $0<y<1$, $f(0) = 0 < y$
$f(1) = 1^2 = 1 > y$

Take $\lambda = y$, Now $f(0) < y < f(1)$:

IVT $\Rightarrow \exists \xi \in (0,1)$ s.t. $f(\xi) = y$ i.e. $\xi^2 = y \Rightarrow \sqrt{y}$ exists.

Case 4 $y > 1$, i.e. $y \in (1,\infty)$

Choose $a = 1$, $f(a) = f(1) = 1^2 = 1 < y$
$b = y$, $f(b) = f(y) = y^2 > y$ $\because y > 1 \iff y^2 > y$

we have $f(1) < y < f(y)$, take $\lambda = y$;

IVT $\Rightarrow \exists \xi \in (1,y)$ s.t. $f(\xi) = \lambda = y$ i.e. $\xi^2 = y \Rightarrow \sqrt{y}$ exists.

$\because y \in (1,\infty)$ and $\xi \in (1,y) \Rightarrow \xi \in (1,\infty)$

$\therefore \sqrt{y}$ exists $\forall y \geq 0$

Exponential functions

Def $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Theorem The exp func converges absolutely $\forall x \in \mathbb{R}$.

Proof Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| \to 0 < 1$

$a_n = \frac{x^n}{n!}$

$\because \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, Series converges abs by ratio test.

$\boxed{\text{Basic property}}$ $e^{x+y} = e^x \cdot e^y \quad \forall x, y \in \mathbb{R}$

$$e^x \cdot e^y = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) \cdot \left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right)$$

$$\begin{aligned}
= (1 &+ y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots \\
&+ x + xy + \frac{xy^2}{2!} + \frac{xy^3}{3!} + \cdots \\
&+ \frac{x^2}{2!} + \frac{x^2y}{2} + \frac{x^2y^2}{2\cdot 2} + \frac{x^2y^3}{2\cdot 3!} + \cdots \\
&+ \frac{x^3}{3!} + \frac{x^3y}{3!} + \frac{x^3y^2}{3!\cdot 2!} + \frac{x^3y^3}{3!\,3!} + \cdots
\end{aligned}$$

$$= 1 + x + y + \frac{x^2}{2!} + xy + \frac{y^2}{2!} + \frac{x^3}{3!} + \frac{x^2y}{2!} + \frac{xy^2}{2!} + \frac{y^3}{3!} + \cdots$$

$$= 1 + (x+y) + \frac{1}{2!}(x^2 + 2xy + y^2) + \frac{1}{3!}(x^3 + 3x^2y + 3xy^2 + y^3) + \cdots$$

Consider terms with degree N:

$$\frac{x^N}{N!} + \frac{x^{N-1}y^1}{(N-1)!} + \frac{x^{N-2}y^2}{(N-2)!\,2!} + \cdots + \frac{x^{N-r}}{(N-r)!}\frac{y^r}{r!} + \cdots + \frac{xy^{N-1}}{(N-1)!} + \frac{y^N}{N!}$$

$$= \frac{1}{N!}\left(x^N + \frac{N!}{(N-1)!}x^{N-1}y + \frac{N!}{(N-2)!\,2!}x^{N-2}y^2 + \cdots + y^N\right)$$

$$= \frac{1}{N!}\left(\sum_{r=0}^{N} \frac{N!}{(N-r)!\,r!} x^{N-r}y^r\right)$$

$$= \frac{1}{N!}(x+y)^N$$

Now $\sum_{N=0}^{\infty} \frac{(x+y)^N}{N!} = e^{x+y}$

$\therefore e^x \cdot e^y = e^{x+y}$

opetres of e^x

$$e^0 = \lim_{b\to\infty}\left(\cancel{1+}\sum_{n=1}^{b}\frac{0^n}{n!}\right) = 1+\lim_{b\to\infty}\sum_{n=1}$$

$$= 1+\lim_{b\to\infty} 0 = 1+0 = 1$$

1. $e^0 = 1$

   Proof $e^0 \underset{\text{def}}{=} 1+0+\frac{0}{2!}+\frac{0}{3!}+\dots = 1$

2. $e^x \cdot e^{-x} \underset{\text{Basic prop}}{=} e^{x+(-x)} = e^0 \underset{1}{=} 1 \Rightarrow e^{-x} = \frac{1}{e^x}$

   i.e. e^{-x} is an inverse of e^x

   Corollary: $e^x \neq 0$ $\because$ 0 has No inverse. $\forall x \in \mathbb{R}$.

3. $e^x > 0$

   Proof $e^x = e^{\frac{x}{2}+\frac{x}{2}} \underset{\text{Basic prop}}{=} e^{\frac{x}{2}} \cdot e^{\frac{x}{2}} = (e^{\frac{x}{2}})^2 > 0$ $\quad \because e^{\frac{x}{2}} \neq 0$ by 2.

4. $e^x > 1$ given $x > 0$

   Proof $e^x \underset{\text{def}}{=} 1 + \underbrace{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}_{\text{all } > 0} > 1$ obv.

5. e^x is a strictly increasing function. i.e. $e^x > e^y$ if $x > y$

   Proof $e^x = e^{(x-y)+y} \underset{\text{Basic prop}}{=} e^{x-y} \cdot e^y > 1 \cdot e^y = e^y$ $\quad \because x > y \Rightarrow x-y > 0$
   $\Rightarrow e^{x-y} > 1$

6. $e^x \geqslant 1+x$ for $x \geqslant 0$

   Proof $e^x \underset{\text{def}}{=} 1 + x + \underbrace{\frac{x^2}{2!} + \frac{x^3}{3!} + \dots}_{\text{all } \geqslant 0} \geqslant 1+x$

7. $e^x \geqslant 1+x$ for $x \leqslant -1$

   Proof $\left.\begin{array}{l} e^x > 0 \quad \forall x \in \mathbb{R} \text{ by } 3 \\ x \leqslant -1 \Rightarrow x+1 \leqslant 0 \end{array}\right\}$ transitivity $e^x \geqslant 1+x$ for $x \leqslant -1$

8. $e^x \leqslant \frac{1}{1-x}$ for $0 < x < 1$

   Proof $e^x \underset{\text{def}}{=} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots < 1 + x + x^2 + x^3 + \dots \underset{\text{GP}}{=} \frac{1}{1-x}$

9. $e^x \leqslant \frac{1}{1-x}$ for $x \leqslant 0$

   Proof Case 1: $x = 0$ $\quad$ LHS $= e^0 = 1$, RHS $= \frac{1}{1-0} = 1$ obv.

   Case 2: $x < 0$
   $-x > 0$
   $e^{(-x)} \geqslant 1 + (-x)$ by 6 $\quad \Rightarrow e^{-x} \geqslant 1-x$
   $e^x \leqslant \frac{1}{1-x}$ QED

$e^x \geq 1+x$ for $-1 < x < 0$

Proof $-1 < x < 0$ by 8: $e^{(-x)} \leq \frac{1}{1-(-x)}$

$\Rightarrow 1 > -x > 0$

$\frac{1}{e^x} \leq \frac{1}{1+x}$

$e^x \geq 1+x$

Remark: $e^x \geq 1+x \quad \forall x \in \mathbb{R}$ by 6, 7, 10.

To prove e^x is continuous $\forall x \in \mathbb{R}$.

11. e^x is continuous at 0, i.e. $\lim_{x\to 0} e^x = e^0 = 1$

Proof $1+x \leq e^x \leq \frac{1}{1-x}$ for x close to 0

$\lim_{x\to 0}(1+x) \leq \lim_{x\to 0} e^x \leq \lim_{x\to 0}\left(\frac{1}{1-x}\right)$

$1 \leq \lim_{x\to 0} e^x \leq 1 \quad \Rightarrow \lim_{x\to 0} e^x = 1 = e^0$ by sandwich theorem.

$\therefore e^x$ is cts at 0 //

12. e^x is continuous at "a" $\forall a \in \mathbb{R}$. i.e. $\lim_{x\to a} e^x = e^a \quad \forall a \in \mathbb{R}$.

$e^x = e^{x-a+a} \underset{\text{Basic prop}}{=} e^{x-a} \cdot e^a \xrightarrow[\substack{x\to a \\ x-a\to 0 \\ \text{by } 11.}]{} 1 \cdot e^a = e^a$

$\therefore e^x$ is cts $\forall x \in \mathbb{R}$ //

13. The range of e^x is $(0,\infty)$.

Case 1 Intermediate Value Theorem

$\lambda > 1$ take $a = 0$ $\quad e^a = e^0 = 1 < \lambda$

take $b = \lambda$ $\quad e^b = e^\lambda \geq 1+\lambda > \lambda$

$\because e^a < \lambda < e^b$, IVT $\Rightarrow \exists \xi \in [a,b] = [0,\lambda]$ s.t. $f(\xi) = e^\xi = \lambda$ for $\lambda > 1$.

$\therefore$ range of e^x at least $(1,\infty)$

Case 2 $\lambda = 1$ obv $\because e^0 = 1$

Case 3 $0 < \lambda < 1 \Rightarrow \frac{1}{\lambda} > 1$

By case 1 $\exists \xi \in [0, \frac{1}{\lambda}]$ s.t. $f(\xi) = e^\xi = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{e^\xi} \underset{\text{by } 2.}{=} e^{-\xi}$

$\because e^x$ is strictly increasing, it is injective

$\Rightarrow \forall x \in (0,\infty)$, $\exists$ a unique ξ, $e^\xi = \lambda$.

Recall $x_n = (1+\frac{1}{n})^n$ increasing, $y_n = (1+\frac{1}{n})^{n+1}$ decreasing $\Big\}$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$

Proposition $\lim x_n = \lim y_n = e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Proof By 6. $e^x \geq 1+x \quad \forall x \in \mathbb{R}$

take $x = \frac{1}{n}$: $e^{\frac{1}{n}} \geq 1+\frac{1}{n}$

$$(e^{\frac{1}{n}})^n \geq (1+\frac{1}{n})^n = x_n$$

$$e \geq x_n$$

$$\lim_{n\to\infty} e = e \geq \lim_{n\to\infty} x_n \quad \text{——} \ (1)$$

take $x = \frac{-1}{n}$ $\quad e^{\frac{-1}{n}} \geq 1-\frac{1}{n}$

$$e^{-1} \geq (1-\frac{1}{n})^n$$

$$e \leq \frac{1}{(1-\frac{1}{n})^n} = \left(\frac{1}{\frac{n-1}{n}}\right)^n = \left(\frac{n}{n-1}\right)^n = \left(1+\frac{1}{n-1}\right)^n = y_{n-1}$$

$$\lim e \leq \lim y_{n-1} = \lim y_n = \lim x_n$$

$$e \leq \lim x_n \quad \text{——} \ (2)$$

Combine (1), (2) : $\lim x_n = \lim y_n = e$ // QED.

Define $f(x) = e^x$, then $f^{-1}(x) = \ln x$

* we define $\ln(y)$ to be the unique x s.t. $e^x = y$ $(y>0)$.

Property of log : $\ln x \leq x-1$

Proof $x = e^{\ln x} \geq 1 + \ln x$

$$x-1 \geq \ln x \quad // \ \text{QED}.$$

Basic property of log

$$\ln(x\cdot y) = \ln x + \ln y$$

Proof $x\cdot y = e^{\ln(xy)} = xy$

$$= (e^{\ln x})(e^{\ln y})$$

$$e^{\ln(xy)} = e^{\ln x + \ln y}$$

$$\ln(xy) = \ln x + \ln y \quad \because e^x \text{ is injective.}$$

Alternate proof of AM-GM

Given $x_1, x_2, \cdots, x_n \geq 0$; $\dfrac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots \cdots x_n}$

Proof Assume $x_i > 0$ for $i = 1, 2, \cdots, n$

let $m = AM = \dfrac{x_1+x_2+\cdots+x_n}{n}$

Apply $\ln x \leq x-1$ for $x = \frac{x_1}{m}, \frac{x_2}{m}, \frac{x_3}{m}, \cdots, \frac{x_n}{m}$

$$\left.\begin{aligned} \ln \frac{x_1}{m} &\leq \frac{x_1}{m} - 1 \\ \ln \frac{x_2}{m} &\leq \frac{x_2}{m} - 1 \\ &\vdots \\ \ln \frac{x_n}{m} &\leq \frac{x_n}{m} - 1 \end{aligned}\right\} + \Rightarrow \ln \frac{x_1}{m} + \ln \frac{x_2}{m} + \ln \frac{x_3}{m} + \cdots + \ln \frac{x_n}{m}$$

$$\leq \frac{x_1+x_2+\cdots+x_n}{m} - n$$

$$= \frac{x_1+x_2+\cdots+x_n}{\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)} - n$$

$$= n - n = 0$$

$$\Rightarrow \ln \frac{x_1}{m} + \ln \frac{x_2}{m} + \cdots \ln \frac{x_n}{m} \leq 0$$

$$\ln\left(\frac{x_1}{m} \cdot \frac{x_2}{m} \cdot \cdots \cdot \frac{x_n}{m}\right) \leq 0 \quad \Big\} \text{ Basic Prop of log !}$$

$$\frac{x_1 \cdot x_2 \cdot \cdots \cdot x_n}{m^n} \leq 1$$

$$x_1 \cdot x_2 \cdots x_n \leq m^n$$

$$\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \leq m = \frac{x_1+x_2+\cdots+x_n}{n}$$

$$\Rightarrow \quad GM \leq AM \qquad \text{QED.}$$

Define a^x $\forall x \in \mathbb{R}$ and $a>0$

$$a^x = e^{x \ln a} \quad \because \quad a = e^{\ln a} \Leftrightarrow a^x = (e^{\ln a})^x \underset{\text{Basic prop}}{=} e^{x \ln a}$$

and: $\log_a(y) = \dfrac{\ln y}{\ln a}$, $y > 0$, $a > 0$.

$\ln x$ is continuous $\forall x>0$

Proof. Let $f:[a,b]\to\mathbb{R}$ be continuous and strictly increasing. then $f:[a,b]\to[f(a),f(b)]$ is bijective.

We can then define $f^{-1}:[f(a),f(b)]\to[a,b]$

$$f^{-1}(y)=x \iff y=f(x).$$

let $x_0=f^{-1}(y_0)$, $y_0=f(x_0)$

Given $\varepsilon>0$, take $\delta=\min\left\{\begin{matrix} f(x_0)-f(x_0-\varepsilon) \\ f(x_0+\varepsilon)-f(x_0)\end{matrix}\right\}>0$

$\because$ f is increasing, $f(x_0)>f(x_0-\varepsilon)$, $f(x_0+\varepsilon)>f(x_0)$, $\therefore \delta>0$.

For $|y-y_0|<\delta$

$\iff y_0-\delta<y<y_0+\delta$

$\iff f(x_0)-\delta<y<f(x_0)+\delta$

$\Rightarrow f(x_0-\varepsilon)<y<f(x_0+\varepsilon)$

$f^{-1}f(x_0-\varepsilon)<f^{-1}(y)<f^{-1}f(x_0+\varepsilon)$

$x_0-\varepsilon<f^{-1}(y)<x_0+\varepsilon$

$f^{-1}(y_0)-\varepsilon<f^{-1}(y)<f^{-1}(y_0)+\varepsilon$

$|f^{-1}(y)-f^{-1}(y_0)|<\varepsilon$

$\delta=\min\left\{\begin{matrix} f(x_0)-f(x_0-\varepsilon) \\ f(x_0+\varepsilon)-f(x_0)\end{matrix}\right\}$

$\Rightarrow \delta\leq f(x_0)-f(x_0-\varepsilon)$

AND $\delta\leq f(x_0+\varepsilon)-f(x_0)$

$\iff f(x_0-\varepsilon)\leq f(x_0)-\delta$

AND $f(x_0)+\delta\leq f(x_0+\varepsilon)$

Now $\forall\varepsilon>0$, $\exists\delta>0$ s.t. $|y-y_0|<\delta\Rightarrow|f^{-1}(y)-f^{-1}(y_0)|<\varepsilon$,

i.e. $\lim_{y\to y_0} f^{-1}(y)=f^{-1}(y_0)$ $\forall y_0\in y:[f(a),f(b)]$

Therefore $f^{-1}(y)$ is continuous $*$ (on $[f(a),f(b)]$).

Let $f(x)=e^x$, where $y=e^x\Rightarrow y=f(x)$;

$\ln y=x$, $f^{-1}(y)=x=\ln y$

Hence by $*$, $\ln(y)$ is continuous on the range $(0,\infty)$ or for $y>0$

$\iff \ln x$ is cts for $x>0$. QED.

Def f is called uniformly continuous on interval I if:

$$\left[\forall \varepsilon>0 : \exists \delta>0 \text{ s.t. } \forall x_1, x_2 \in I : |x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)| < \varepsilon\right]$$

Remark 1 δ does NOT depend on x_2

Remark 2 Compare def with f is cts on I:

$$\forall x_2 \in I : \forall \varepsilon>0 \ \exists \delta(x_2)>0 \text{ s.t. } |x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|<\varepsilon$$

Remark 3 $f(x)=x^2$ is NOT uniformly cts on $[0,\infty)$

Proof NOT uniformly cts means:

$$\exists \varepsilon>0 \text{ s.t. } \forall \delta>0, \exists x_1,x_2 \in I \text{ s.t. } \underbrace{|x_1-x_2|<\delta}_{A} \wedge \underbrace{|f(x_1)-f(x_2)| > \varepsilon}_{B}$$

Need to satisfy A & B.

* Take $x_2 = x_1 + \frac{\delta}{2} \Rightarrow |x_1-x_2| = \frac{\delta}{2} < \delta$

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| \\
&= |(x_1-x_2)(x_1+x_2)| \\
&= |x_1-x_2|\,(x_1+x_2) \\
&= \frac{\delta}{2}\cdot\left(x_1 + x_1 + \frac{\delta}{2}\right) \\
&> \frac{\delta}{2}\cdot(2x_1) \\
&= \delta \cdot x_1 \\
&= \varepsilon
\end{aligned}
$$

* Take $x_1 = \frac{\varepsilon}{\delta} \Leftrightarrow \delta x_1 = \varepsilon$
* Allow ε be any $\mathbb{R}>0$.

$\therefore$ By taking $x_1 = \frac{\varepsilon}{\delta}$ & $x_2 = x_1 + \frac{\delta}{2}$;

$\forall \varepsilon>0$ we have $|x_1-x_2|<\delta$ satisfied and $|f(x_1)-f(x_2)|>\varepsilon$ satisfied $\forall \delta>0$ // QED.

Theorem

let f be cts on the compact interval $[a,b]$;

then f is uniformly cts on $[a,b]$.